EXISTENCE OF BOUNDEN NONOSCILLATORY SOLUTIONS OF CERTAIN NONLINEAR NEUTRAL DELAY DIFFERENCE EQUATIONS OF SECOND ORDER

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ABSTRACT

Non oscillation of a class of nonlinear neutral delay difference equations with positive and negative coefficients of the form

$$\Delta^2\left[y(n)+p(n)y(n-m)\right]+f_1(n)G_1(y(n-k_1))-f_2(n)G_2(y(n-k_2))=f(n) \qquad \text{(E)}$$

is studied. We obtain the sufficient conditions for the existence of a non-oscillatory solutions of (E) under the assumption

$$\sum_{n=0}^{\infty} f_i(\text{n})<\infty, \quad \text{for } \text{i}=1,2$$

for ranges of $p(n)=1, p(n)=-1, 1<p(n)<\infty$ & $-\infty<p(n)<-1.$

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1. INTRODUCTION:

In this paper we study non oscillation of a class of neutral delay difference equation with positive and negative coefficients of the form

$$\Delta^2[y(n)+p(n)y(n-m)]+f_1(n)G_1(y(n-k_1))-f_2(n)G_2(y(n-k_2))=f(n). \qquad (1)$$

where $p(n),\ f_1(n),\ f_2(n)$ are real valued functions defined on $N(n_0)=\{n_0,n_0+1,n_0+2,n_0+3............\}$, $n_0\geq 0$ such that $f_1(n)\geq 0$, $f_2(n)\geq 0$, G_1,G_2 are continuous real valued functions. G_1 and G_2 are non decreasing and $xG_i(x)>0$ for i=1, 2, $x\neq 0,\ n>0$ and $k_1,k_2,m\geq 0$ are integers, Δ is forward difference operator defined by equation, $\Delta x(n)=x(n+1)-x(n)$.

Qualitative behavior of solution of differential equations / difference equations is a subject to investigators. We refer to [3], and the work of [2], [1] and the references therein.The corresponding differential equation to the difference equation (1) can be written as

$$\frac{d^2}{dt^2}[y(t)+p(t)y(t-\tau)]+f_1(t)G_1(y(t-\sigma_1))-f_2(t)G_2(y(t-\sigma_2))=f(n). \qquad (2)$$

It is to remark that this equation when $f_2(t)\equiv 0, p(t)\equiv 0$ becomes a second order delay differential equation and we find numerous results regarding the solutions of this equations. Several researchers discussed non oscillation and asymptotic behavior of solution of delay and neutral difference equations of second order. A close observation reveals that the study of difference equation is more or less similar to that of a differential equation. (See [4], [6], [8], [9] and [9]). In this recent paper [8,9]. Parhi and Tripathy discussed oscillation and asymptotic behavior of solution of the equation

$$\Delta[y(n)-y(n-m)]+f_1(n)G_1(y(n-k_1))=0, \qquad (3)$$

when $f_1(n)<0$ or when $f_1(n)>0$ under the condition

$$\sum_{n=0}^{\infty} f_1(n)=\infty.$$

It is predicted that the oscillation properties are not restricted to the sign of f_1. The motivation of the present work comes under two directions. Firstly due to the above prediction and next due to the work in [5].

Where the authors considered the linear neutral differential equation

$$\frac{d^2}{dt^2}\left(y(t)+py(t-\tau)\right)+f_1(t)y(t-\sigma_1)-f_2(t)y(t-\sigma_2)=f(n), \qquad (4)$$

where $p \neq -1$ is constant. The discrete analogue of equation (4) is a particular case of our equation (1).When $f(n)=0$ the existence of non oscillatory solutions was discussed in [11]. And when $0<p(n)<1, -1<p(n)<0, -1<p(n)<1$ the oscillatory solutions was discussed in[7]. In this present paper we discussed the non oscillation behavior of the solution of some more ranges of $p(n)$.

Let $\rho=\max\{n,k_1,k_2\}$ and n_0 be a fixed non negative integer.

By a solution of equation (1), we mean a real sequence $y(n)$ which is defined for all positive integers $n \geq n_0-\rho$ and satisfies the equation (1) for $n \geq n_0$.

If the initial condition

$$y(n)=A_n \quad for \;\; n_0-\rho \leq n \leq n_0 \;, \qquad (5)$$

are given, then equation (1) has a unique solution satisfying the initial condition (5).

As is customary, a solution of (1) is said to be oscillate if for every integer $N>0$, there exists and $n \geq N$ such that $y(n)y(n+1) \leq 0$. Otherwise, the solution is called no oscillatory.

MAIN RESULT:

We need the following the hypotheses in our discussion:

(H_1) : $G_i \in C(R,R)$, G_i is non decreasing for $i=1,2$.

(H_2) : $xG_i(x)>0$ for $x \neq 0$, $i=1,2$.

(H_3) : G_i, $i=1,2$ is Lipschitizain on the interval of the type [a , b], $0<a<b<\infty$.

(H_4) : $\sum_{n=0}^{\infty} f_i(n)<\infty$ for $i=1,2$.

Now we have the following main theorem.

THEOREM 2.1: Suppose $p(n) = -1$ and $(H_1)-(H_4)$ hold. If $\sum_{n=0}^{\infty} f(n) < \infty$. Then there exists abounded non oscillatory solution of the equation (1).

Proof: Let $0 < b_1 < 1$ be such that $b_1 \neq \frac{1}{2}$.

From hypotheses, we can find N_1 sufficiently large such that

$$M_1 \sum_{n=N_1}^{\infty} f_1(n) < \frac{1-2b_1}{20},$$

$$M_2 \sum_{n=N_1}^{\infty} f_2(n) < \frac{1-2b_1}{10},$$

$$\sum_{n=N_1}^{\infty} f(n) < \frac{1-2b_1}{20},$$

where $M_1 = \max\{ L_1, G_1(b_1) \}$, $M_2 = \max\{ L_2, G_2(b_1) \}$ and L_1, L_2 are Lipschitz constants of G_1 and G_2 respectively on $\left[\frac{1-b_1}{20}, b_1\right]$.

Let $X = l_\infty^{N_1}$ be the Banach space of all real valued functions $x(n)$, $n \geq N_1$ with supremum norm $\| x \| = \sup\{ | x(n) | : n \geq N_1 \}$.

Define

$$S = \left\{ x \in X : \frac{1-b_1}{20} \leq x(n) \leq b_1 ,\ n \geq N_1 \right\},$$

we see that S is a complete metric space with the metric induced by the norm on X.

For $y \in S$, define

$Ty(n) = Ty(N_1 + \rho)$, $\quad N_1 \leq n \leq N_1 + \rho$

$$= y(n-m) + \frac{1-b_1}{10} - \sum_{s=n}^{\infty} f_1(s) G_1(y(s-k_1)) + \sum_{s=n}^{\infty} f_2(s) G_2(y(s-k_2)) + \sum_{s=n}^{\infty} f(s), \quad n \geq N_1 + \rho.$$

Clearly,

$$Ty(n) < b_1 + \frac{1-2b_1}{10} + \frac{1-b_1}{10} + \frac{1-2b_1}{20}$$

$$< \frac{5+12b_1}{20} < b_1,$$

and

$$Ty(n) \geq \frac{1-b_1}{10} - \frac{1-2b_1}{20}$$

$$= \frac{1}{20} > \frac{1-b_1}{20}.$$

This implies that $Ty \in S$ and therefore $T : s \to s$.

Also, for $x, y \in S$,

$$|Ty(n) - Tx(n)| \leq \|y-x\| + \frac{1-2b_1}{20}\|y-x\| + \frac{1-2b_1}{10}\|y-x\|$$

$$= \frac{23-6b_1}{20}\|y-x\|,$$

that is,

$$|Ty(n) - Tx(n)| < \frac{23-6b_1}{20}\|y-x\| .$$

Hence T is a contraction. By the contraction principle, T has a unique fixed point $y(n)$ in the interval $\left[\frac{1-b_1}{20}, b_1\right]$, which is therefore the required non oscillatory solution of the equation (1).

THEOREM 2.2: Suppose $p(n) = 1.$ and $(H_1)-(H_4)$ hold. If $\sum_{n=0}^{\infty} f(n) < \infty$. Then there exists abounded non oscillatory solution of the equation (1).

Proof: We can choose a possible a positive integer N_1 such that

$$M_1 \sum_{n=N_1}^{\infty} f_1(n) < \frac{1-b_1}{10},$$

$$M_2 \sum_{n=N_1}^{\infty} f_2(n) < \frac{1-b_1}{5},$$

$$\sum_{n=N_1}^{\infty} f(n) < \frac{1-b_1}{5},$$

where $M_1 = \max\{L_1, G_1(b_1)\}$, $M_2 = \max\{L_2, G_2(b_1)\}$ and L_1 L_2, are Lipschitz constants of G_1 and G_2 on $\left[\frac{b_1-1}{10}, 1\right]$ respectively.

Let $X = l_\infty^{N_1}$ be the Banach space of all real valued functions $x(n)$, $n \geq N_1$ with suprmum norm

$$\| x \| = \sup\{| x(n) |: n \geq N_1\}.$$

We define,

$$S = \left\{ x \in X : \frac{b_1-1}{10} \leq x(n) \leq 1, \ \ n \geq N_1 \right\}.$$

It is easy to see that S is a complete metric space, where the metric is induced by norm on X.

For $y \in S$, define

$$Ty(n) = Ty(N_1+\rho), \quad N_1 \leq n \leq N_1+\rho$$

$$= -p(n)y(n-m) + \frac{1+9b_1}{10} - \sum_{s=n}^{\infty} f_1(s)G_1(y(s-k_1))$$

$$+ \sum_{s=n}^{\infty} f_2(s)G_2(y(s-k_2)) + \sum_{s=n}^{\infty} f(s), \qquad n \geq N_1+\rho.$$

In view of hypotheses, we observe that

$$Ty(n) < \frac{1+9b_1}{10} + M_2 \sum_{s=n}^{\infty} f_2(s) + \sum_{s=n}^{\infty} f(s)$$

$$< \frac{1+9b_1}{10} + \frac{1-b_1}{10} + \frac{1-b_1}{5}$$

$$= \frac{2-3b_1}{5} < 1,$$

and

$$Ty(n) > -b_1 + \frac{1+9b_1}{10} - M_1 \sum_{s=n}^{\infty} f_1(s)$$

$$> -b_1 + \frac{1+9b_1}{10} - \frac{1-b_1}{5} = \frac{b_1-1}{10}, \qquad \text{for } n \geq N_1 + \rho .$$

Consequently, $Ty \in S$, that is $T : S \to S$.

Further for $x \in S$, consider

$$|\, Ty(n) - Tx(n) \,| \leq b_1 \|y-x\| + \frac{1-b_1}{10}\|y-x\| + \frac{1-b_1}{5}\|y-x\|$$

$$= \left(b_1 + \frac{1-b_1}{10} + \frac{2-2b_1}{5}\right)\|y-x\|$$

$$= \frac{7b_1+3}{10}\|y-x\|.$$

Thus $\quad |\, Ty(n) - Tx(n) \,| \leq \frac{7b_1+3}{10}\|y-x\|$ for every $x, y \in S$.

Hence T is contraction. Consequently, T has a unique fixed point y in S. which is solution of equation (1) in the interval $\left[\frac{b_1-1}{10}, 1\right]$ and is the bounded non oscillatory solution.

THEOREM 2.3: Suppose $1 < b_1 \leq p(n) \leq b_2 < \infty$ and $(H_1)-(H_4)$ hold. If $\sum_{n=0}^{\infty} f(n) < \infty$. Then the equation (1) admits a non oscillatory and bounded solution.

Proof: Let $1 < b_1 \leq p(n) \leq b_2 < \infty$ be such that $b_1 > 0$, $b_2 > 0$ and $b_2 < 1 + 3b_1$. As earlier, from the hypotheses, it is possible to choose a positive integer N_1 large enough such that

$$\mathrm{M}_1 \sum_{n=N_1}^{\infty} f_1(n) < \frac{b_2-1}{4} - b_1,$$

$$\mathrm{M}_2 \sum_{n=N_1}^{\infty} f_2(n) < \frac{b_2-1}{4},$$

$$\sum_{n=N_1}^{\infty} f(n) < \frac{b_2 - 1}{4},$$

where M_1, M_2 and N_1 are same as in earlier case on the interval $\left[\frac{b_2-1}{2}, 1\right]$.

Let $X = l_\infty^{N_1}$ be the Banach space of all real valued functions $x(n)$, $n \geq N_1$ with supremum norm $\| x \| = \sup \{ | x(n) | : n \geq N_1 \}$.

Let $S = \left\{ x \in X : \frac{b_2 - 1}{2} \leq x(n) \leq \infty, \ n \geq N_1 \right\}$.

Define a mapping T as

$$Ty(n) = Ty(N_1 + \rho), \quad N_1 \leq n \leq N_1 + \rho$$

$$= -p(n)y(n-m) + \frac{3(b_2-1)}{4} - \sum_{s=n}^{\infty} f_1(s) G_1(y(s-k_1))$$

$$+ \sum_{s=n}^{\infty} f_2(s) G_2(y(s-k_2)) + \sum_{s=n}^{\infty} f(s), \quad n \geq N_1 + \rho.$$

For $y \in S$ and $n \geq N_1 + \rho$,

$$Ty(n) \leq -b_2 + \frac{3(b_2-1)}{4} + M_2 \sum_{s=n}^{\infty} f_2(s) + \sum_{s=n}^{\infty} f(n)$$

$$< -b_2 + \frac{3(b_2-1)}{4} + \frac{b_2-1}{4} + \frac{b_2-1}{4}$$

$$= \frac{b_2-5}{4} < \infty,$$

and

$$Ty(n) \geq -b_2 + \frac{3(b_2-1)}{4} - \left(\frac{b_2-1}{4} - b_1 \right)$$

$$\geq \frac{b_2-1}{2}.$$

Consequently, $Ty \in S$, that is $T : S \to S$.

Further for $x \in S$, consider

$$|\,Ty(n) - Tx(n)\,| \le b_2 \|y - x\| + \left(\frac{b_2 - 1}{4} - b_1 \right) \|y - x\| + \frac{b_2 - 1}{4} \|y - x\|$$

$$\le \left(b_2 + \frac{b_2 - 1}{4} - b_1 + \frac{b_2 - 1}{4} \right) \|y - x\|$$

$$\le \left(\frac{6b_2 - 2}{4} - b_1 \right) \|y - x\|$$

$$\le \left(\frac{14b_1 + 4}{4} \right) \|y - x\|$$

Thus $|\,Ty(n) - Tx(n)\,| \le \frac{7b_1 + 2}{2} \|y - x\|$ for every $x, y \in S$.

Hence T is contraction. Consequently, T has a unique fixed point y in S. which is solution of equation (1) in the interval $\left[\frac{b_2 - 1}{2}, \infty \right]$; and is the bounded non oscillatory solution.

THEOREM 2.4: Suppose $-\infty < b_2 \le p(n) \le b_1 < -1$ and $(H_1) - (H_4)$ hold. If $\sum_{n=0}^{\infty} f(n) < \infty$. Then the equation (1) admits a non oscillatory and bounded solution.

Proof: Let $-\infty < b_2 \le p(n) \le b_1 < -1$ be such that $b_1 < 0, b_2 < 0$ and $b_1 < 1 + 3b_2$. As earlier, from the hypotheses, it is possible to choose a positive integer N_1 large enough such that

$$M_1 \sum_{n=N_1}^{\infty} f_1(n) < \frac{b_1 + 1}{4} - b_2,$$

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$$M_2 \sum_{n=N_1}^{\infty} f_2(n) < \frac{b_1 + 1}{4},$$

$$\sum_{n=N_1}^{\infty} f(n) < \frac{b_1 + 1}{4},$$

where M_1, M_2 and N_1 are same as in earlier case on the interval $\left[\frac{b_1-1}{2}, 1\right]$.

Let $X = l_\infty^{N_1}$ be the Banach space of all real valued functions $x(n)$, $n \geq N_1$ with supremum norm

$$\| x \| = \sup \{ | x(n) | : n \geq N_1 \}.$$

Let

$$S = \left\{ x \in X : \frac{b_2+1}{2} \leq x(n) \leq 1, \quad n \geq N_1 \right\}.$$

Define a mapping T as

$$Ty(n) = Ty(N_1+\rho), \quad N_1 \leq n \leq N_1+\rho$$

$$= -p(n)y(n-m) + \frac{3(b_1-1)}{4} - \sum_{s=n}^{\infty} f_1(s)G_1(y(s-k_1))$$

$$+ \sum_{s=n}^{\infty} f_2(s)G_2(y(s-k_2)) + \sum_{s=n}^{\infty} f(s), \quad n \geq N_1+\rho.$$

For $y \in S$ and $n \geq N_1 + \rho$,

$$Ty(n) \leq -b_1 + \frac{3(b_1-1)}{4} + M_2 \sum_{s=n}^{\infty} f_2(s) + \sum_{s=n}^{\infty} f(n)$$

$$< -b_1 + \frac{3(b_1-1)}{4} + \frac{b_1+1}{4} + \frac{b_1+1}{4}$$

$$= \frac{b_1-1}{4} < 1,$$

and

$$Ty(n) \geq -b_2 + \frac{3(b_1-1)}{4} - \left(\frac{b_1+1}{4} - b_2\right)$$

$$\geq \frac{b_1-1}{2}.$$

Consequently, $Ty \in S$, that is $T : S \to S$.

Further for $x \in S$, consider

$$|Ty(n) - Tx(n)| \le b_1\|y-x\| + \left(\frac{b_1+1}{4} - b_2\right)\|y-x\| + \frac{b_1+1}{4}\|y-x\|$$

$$\le \left(b_1 + \frac{b_1+1}{4} - b_2 + \frac{b_1-1}{4}\right)\|y-x\|$$

$$\le \left(\frac{6b_1+2}{4} - b_2\right)\|y-x\|$$

$$\le \left(\frac{14b_2+8}{4}\right)\|y-x\|$$

Thus $|Ty(n) - Tx(n)| \le \frac{7b_2+4}{2}\|y-x\|$ for every $x, y \in S$.

Hence T is contraction. Consequently, T has a unique fixed point y in S. which is solution of equation (1) in the interval $\left[\frac{b_1-1}{2}, 1\right]$; and is the bounded non oscillatory solution.

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