SOME RESULTS ON THE GROWTH OF ENTIRE FUNCTIONS ON THE BASIS OF CENTRAL INDEX

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ABSTRACT:

In this paper we study the comparative growth properties related to order (lower order) and hyper order (hyper lower order) of entire functions on the basis of central index.

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Introduction, Definitions and Notations:

Let

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

be an entire function. $M(r,f) = \max_{|z|=r}|f(z)|$ denote the maximum modulus of f on $|z| = r$ and $\mu(r,f) = \max_{n\geq 0}|a_n|r^n$ denote the maximum term of f on $|z| = r$. The central index $\nu_f(r)$ is the greatest exponent m such that $|a_m|r^m = \mu(r,f)$. We note that $\nu_f(r)$ is real, non-decreasing function of r.

We do not explain the standard definitions and notations in the theory of entire function as those are available in [3]. In the sequel the following two notations are used:

$$\log^{[k]} x = \log(\log^{[k-1]} x) \qquad for\ k = 1,2,3,\ldots$$

$$\text{and} \quad \log^{[0]} x = x$$

and

$$\exp^{[k]} x = \exp(\exp^{[k-1]} x) \qquad for\ k = 1,2,3,\ldots$$

$$\text{and} \quad \exp^{[0]} x = x$$

Definition 1: [2] The order ρ_f of an entire function f is defined as

$$\rho_f = \limsup_{r\to\infty} \frac{\log \nu_f(r)}{\log r}$$

The lower order λ_f of an entire function f is defined as

$$\lambda_f = \liminf_{r\to\infty} \frac{\log \nu_f(r)}{\log r}$$

We say that f is of regular growth if $\rho_f = \lambda_f$.

Definition 2: [1] The hyper order $\bar{\rho}_f$ of an entire function f is defined as

$$\bar{\rho}_f = \limsup_{r\to\infty} \frac{\log^{[2]} \nu_f(r)}{\log r}$$

The hyper lower order $\bar{\lambda}_f$ of an entire function f is defined as

$$\bar{\lambda}_f = \liminf_{r\to\infty} \frac{\log^{[2]} \nu_f(r)}{\log r}$$

In this paper we study the comparative growth properties related to order (lower order) and hyper order (hyper lower order) of entire functions on the basis of central index.

Theorems.

In this section we present the main results of the paper.

Theorem 1: Let f and g be two entire functions. Also let $0 < \lambda_{fog} \leq \rho_{fog} < \infty$ and $0 < \lambda_g \leq \rho_g < \infty$. Then

$$\frac{\lambda_{fog}}{\rho_g} \leq \liminf_{r\to\infty} \frac{\log \nu_{fog}(r)}{\log \nu_g(r)} \leq \min\left\{\frac{\lambda_{fog}}{\lambda_g}, \frac{\rho_{fog}}{\rho_g}\right\} \leq \max\left\{\frac{\lambda_{fog}}{\lambda_g}, \frac{\rho_{fog}}{\rho_g}\right\}$$

$$\leq \limsup_{r\to\infty} \frac{\log \nu_{fog}(r)}{\log \nu_g(r)} \leq \frac{\rho_{fog}}{\lambda_g}$$

Proof: From the definition of order and lower order of an entire function g, we have for arbitrary positive ε and for all sufficiently large values of r that

$$\log \nu_g(r) \leq (\rho_g + \varepsilon) \log r \qquad (1)$$

and

$$\log \nu_g(r) \geq (\lambda_g - \varepsilon) \log r \qquad (2)$$

Also for a sequence of values of r tending to infinity,

$$\log \nu_g(r) \leq (\lambda_g + \varepsilon) \log r \qquad (3)$$

and

$$\log \nu_g(r) \geq (\rho_g - \varepsilon) \log r \qquad (4)$$

Again from the definition of order and lower order of the composite entire function fog, we have for arbitrary positive ε and for all sufficiently large values of r,

$$\log \nu_{fog}(r) \leq (\rho_{fog} + \varepsilon) \log r \qquad (5)$$

and

$$\log \nu_{fog}(r) \geq (\lambda_{fog} - \varepsilon) \log r \qquad (6)$$

Again for a sequence of values of r tending to infinity

$$\log \nu_{fog}(r) \leq (\lambda_{fog} + \varepsilon) \log r \qquad (7)$$

and

$$\log \nu_{fog}(r) \geq (\rho_{fog} - \varepsilon) \log r \tag{8}$$

Now from (1) and (6) it follows for all sufficiently large values of r that

$$\frac{\log \nu_{fog}(r)}{\log \nu_g(r)} \geq \frac{\lambda_{fog} - \varepsilon}{\rho_g + \varepsilon}$$

As $\varepsilon(> 0)$ is arbitrary, we obtain that

$$\liminf_{r\to\infty} \frac{\log \nu_{fog}(r)}{\log \nu_g(r)} \geq \frac{\lambda_{fog}}{\rho_g} \tag{9}$$

Again, combining (2) and (7) we get for a sequence of values of r tending to infinity

$$\frac{\log \nu_{fog}(r)}{\log \nu_g(r)} \leq \frac{\lambda_{fog} + \varepsilon}{\lambda_g - \varepsilon}$$

Since $\varepsilon(> 0)$ is arbitrary, it follows that

$$\liminf_{r\to\infty} \frac{\log \nu_{fog}(r)}{\log \nu_g(r)} \leq \frac{\lambda_{fog}}{\lambda_g} \tag{10}$$

Similarly, from (4) and (5) it follows for a sequence of values of r tending to infinity that

$$\frac{\log \nu_{fog}(r)}{\log \nu_g(r)} \leq \frac{\rho_{fog} + \varepsilon}{\rho_g - \varepsilon}$$

As $\varepsilon(> 0)$ is arbitrary, we obtain that

$$\liminf_{r\to\infty} \frac{\log \nu_{fog}(r)}{\log \nu_g(r)} \leq \frac{\rho_{fog}}{\rho_g} \tag{11}$$

Now combining (9), (10) and (11) we get that

$$\frac{\lambda_{fog}}{\rho_g} \leq \liminf_{r\to\infty} \frac{\log \nu_{fog}(r)}{\log \nu_g(r)} \leq min\left\{\frac{\lambda_{fog}}{\lambda_g}, \frac{\rho_{fog}}{\rho_g}\right\} \tag{12}$$

Now from (3) and (6) we obtain for a sequence of values of r tending to infinity that

$$\frac{\log \nu_{fog}(r)}{\log \nu_g(r)} \geq \frac{\lambda_{fog} - \varepsilon}{\lambda_g + \varepsilon}$$

Choosing $\varepsilon \to 0$ we get that

$$\limsup_{r\to\infty} \frac{\log \nu_{fog}(r)}{\log \nu_g(r)} \geq \frac{\lambda_{fog}}{\lambda_g} \tag{13}$$

Again from (2) and (5) it follows for all sufficiently large values of r that

$$\frac{\log \nu_{fog}(r)}{\log \nu_g(r)} \leq \frac{\rho_{fog} + \varepsilon}{\lambda_g - \varepsilon}$$

As $\varepsilon(> 0)$ is arbitrary, we obtain that

$$\limsup_{r\to\infty} \frac{\log \nu_{fog}(r)}{\log \nu_g(r)} \leq \frac{\rho_{fog}}{\lambda_g} \tag{14}$$

Similarly, combining (1) and (8) we get for a sequence of values of r tending to infinity that

$$\frac{\log \nu_{fog}(r)}{\log \nu_g(r)} \geq \frac{\rho_{fog} - \varepsilon}{\rho_g + \varepsilon}$$

Since $\varepsilon(> 0)$ is arbitrary, it follows that

$$\limsup_{r\to\infty} \frac{\log \nu_{fog}(r)}{\log \nu_g(r)} \geq \frac{\rho_{fog}}{\rho_g} \tag{15}$$

Therefore combining (13), (14) and (15) we get that

$$\max\left\{\frac{\lambda_{fog}}{\lambda_g}, \frac{\rho_{fog}}{\rho_g}\right\} \leq \limsup_{r\to\infty} \frac{\log \nu_{fog}(r)}{\log \nu_g(r)} \leq \frac{\rho_{fog}}{\lambda_g} \tag{16}$$

Thus the theorem follows from (12) and (16).

Example 1: Considering $f = z,\ g = \exp z$ one can easily verify that the sign '$\leq$' in Theorem 1 cannot be replaced by '$<$' only.

Remark 1: If we take $0 < \lambda_f \leq \rho_f < \infty$ instead of $0 < \lambda_g \leq \rho_g < \infty$ and the other conditions remain the same then also Theorem 1 holds with g replaced by f in the denominator as we see in the next theorem.

Theorem 2: Let f and g be two entire functions. Also let $0 < \lambda_{fog} \leq \rho_{fog} < \infty$ and $0 < \lambda_f \leq \rho_f < \infty$. Then

$$\frac{\lambda_{fog}}{\rho_f} \leq \liminf_{r\to\infty} \frac{\log \nu_{fog}(r)}{\log \nu_f(r)} \leq \min\left\{\frac{\lambda_{fog}}{\lambda_f}, \frac{\rho_{fog}}{\rho_f}\right\} \leq \max\left\{\frac{\lambda_{fog}}{\lambda_f}, \frac{\rho_{fog}}{\rho_f}\right\}$$

$$\leq \limsup_{r\to\infty} \frac{\log \nu_{fog}(r)}{\log \nu_f(r)} \leq \frac{\rho_{fog}}{\lambda_f}$$

Proof. From the definition of order and lower order of an entire function f, we have for arbitrary positive ε and for all sufficiently large values of r that

$$\log \nu_f(r) \leq (\rho_f + \varepsilon)\log r \quad (17)$$

and

$$\log \nu_f(r) \geq (\lambda_f - \varepsilon)\log r. \quad (18)$$

Also for a sequence of values of r tending to infinity,

$$\log \nu_f(r) \leq (\lambda_f + \varepsilon)\log r \quad (19)$$

and

$$\log \nu_f(r) \geq (\rho_f - \varepsilon)\log r \quad (20)$$

Again from the definition of order and lower order of the composite entire function fog, we have for arbitrary positive ε and for all sufficiently large values of r

$$\log \nu_{fog}(r) \leq (\rho_{fog} + \varepsilon)\log r \quad (21)$$

and

$$\log \nu_{fog}(r) \geq (\lambda_{fog} - \varepsilon)\log r \quad (22)$$

Again, for a sequence of values of r tending to infinity

$$\log \nu_{fog}(r) \leq (\lambda_{fog} + \varepsilon)\log r \quad (23)$$

and

$$\log \nu_{fog}(r) \geq (\rho_{fog} - \varepsilon)\log r \quad (24)$$

Now from (17) and (22) it follows for all sufficiently large values of r that

$$\frac{\log \nu_{fog}(r)}{\log \nu_f(r)} \geq \frac{\lambda_{fog} - \varepsilon}{\rho_f + \varepsilon}$$

As $\varepsilon(> 0)$ is arbitrary, we obtain that

$$\liminf_{r\to\infty} \frac{\log \nu_{fog}(r)}{\log \nu_f(r)} \geq \frac{\lambda_{fog}}{\rho_f} \quad (25)$$

Again, combining (18) and (23) we get for a sequence of values of r tending to infinity

$$\frac{\log \nu_{fog}(r)}{\log \nu_f(r)} \leq \frac{\lambda_{fog} + \varepsilon}{\lambda_f - \varepsilon}$$

Since $\varepsilon(> 0)$ is arbitrary, it follows that

$$\liminf_{r\to\infty}\frac{\log \nu_{fog}(r)}{\log \nu_f(r)} \le \frac{\lambda_{fog}}{\lambda_f} \qquad (26)$$

Similarly, from (20) and (21) it follows for a sequence of values of r tending to infinity that

$$\frac{\log \nu_{fog}(r)}{\log \nu_f(r)} \le \frac{\rho_{fog}+\varepsilon}{\rho_f-\varepsilon}$$

As $\varepsilon(>0)$ is arbitrary, we obtain that

$$\liminf_{r\to\infty}\frac{\log \nu_{fog}(r)}{\log \nu_f(r)} \le \frac{\rho_{fog}}{\rho_f} \qquad (27)$$

Now combining (25), (26) and (27) we get that

$$\frac{\lambda_{fog}}{\rho_f} \le \liminf_{r\to\infty}\frac{\log \nu_{fog}(r)}{\log \nu_f(r)} \le \min\left\{\frac{\lambda_{fog}}{\lambda_f},\frac{\rho_{fog}}{\rho_f}\right\} \qquad (28)$$

Now, from (19) and (22) we obtain for a sequence of values of r tending to infinity

$$\frac{\log \nu_{fog}(r)}{\log \nu_f(r)} \ge \frac{\lambda_{fog}-\varepsilon}{\lambda_f+\varepsilon}$$

Choosing $\varepsilon(>0)$ we get that

$$\limsup_{r\to\infty}\frac{\log \nu_{fog}(r)}{\log \nu_f(r)} \ge \frac{\lambda_{fog}}{\lambda_f} \qquad (29)$$

Again, from (18) and (21) it follows for all sufficiently large values of r

$$\frac{\log \nu_{fog}(r)}{\log \nu_f(r)} \le \frac{\rho_{fog}+\varepsilon}{\lambda_f-\varepsilon}$$

As $\varepsilon(>0)$ is arbitrary, we obtain that

$$\limsup_{r\to\infty}\frac{\log \nu_{fog}(r)}{\log \nu_f(r)} \le \frac{\rho_{fog}}{\lambda_f} \qquad (30)$$

Similarly, combining (17) and (24) we get for a sequence of values of r tending to infinity

$$\frac{\log \nu_{fog}(r)}{\log \nu_f(r)} \ge \frac{\rho_{fog}-\varepsilon}{\rho_f+\varepsilon}$$

Since $\varepsilon(>0)$ is arbitrary, it follows that

$$\limsup_{r\to\infty}\frac{\log \nu_{fog}(r)}{\log \nu_f(r)} \ge \frac{\rho_{fog}}{\rho_f} \qquad (31)$$

Therefore combining (29), (30) and (31) we get that

$$\max\left\{\frac{\lambda_{fog}}{\lambda_f},\frac{\rho_{fog}}{\rho_f}\right\}\leq \limsup_{r\to\infty}\frac{\log \nu_{fog}(r)}{\log \nu_f(r)}\leq \frac{\rho_{fog}}{\lambda_f} \qquad (32)$$

Thus the theorem follows from (28) and (32).

Example 2 : Taking $f=\exp z, g=z$ one can easily verify that the sign '$\leq$' in Theorem 2 cannot be replaced by '$<$'only.

Extending the notion we may prove the subsequent theorems using hyper order (hyper lower order).

Theorem 3: Let f and g be two entire functions. Also let $0<\bar{\lambda}_{fog}\leq \bar{\rho}_{fog}<\infty$ and $0<\bar{\lambda}_g\leq \bar{\rho}_g<\infty$. Then

$$\frac{\bar{\lambda}_{fog}}{\bar{\rho}_g}\leq \liminf_{r\to\infty}\frac{\log^{[2]}\nu_{fog}(r)}{\log^{[2]}\nu_g(r)}\leq \min\left\{\frac{\bar{\lambda}_{fog}}{\bar{\lambda}_g},\frac{\bar{\rho}_{fog}}{\bar{\rho}_g}\right\}\leq \max\left\{\frac{\bar{\lambda}_{fog}}{\bar{\lambda}_g},\frac{\bar{\rho}_{fog}}{\bar{\rho}_g}\right\}$$
$$\leq \limsup_{r\to\infty}\frac{\log^{[2]}\nu_{fog}(r)}{\log^{[2]}\nu_g(r)}\leq \frac{\bar{\rho}_{fog}}{\bar{\lambda}_g}$$

Proof: From the definition of hyper order and hyper lower order of an entire function g we have for arbitrary positive ε and for all sufficiently large values of r that

$$\log^{[2]}\nu_g(r)\leq(\bar{\rho}_g+\varepsilon)\log r \qquad (33)$$

and

$$\log^{[2]}\nu_g(r)\geq(\bar{\lambda}_g-\varepsilon)\log r \qquad (34)$$

Also, for a sequence of values of r tending to infinity

$$\log^{[2]}\nu_g(r)\leq(\bar{\lambda}_g+\varepsilon)\log r \qquad (35)$$

and

$$\log^{[2]}\nu_g(r)\geq(\bar{\rho}_g-\varepsilon)\log r \qquad (36)$$

Again from the definition of hyper order and hyper lower order of the composite entire function fog, we have for arbitrary positive ε and for all sufficiently large values of r

$$\log^{[2]}\nu_{fog}(r)\leq(\bar{\rho}_{fog}+\varepsilon)\log r \qquad (37)$$

and

$$\log^{[2]}\nu_{fog}(r)\geq(\bar{\lambda}_{fog}-\varepsilon)\log r \qquad (38)$$

Again, for a sequence of values of r tending to infinity

$$\log^{[2]} \nu_{fog}(r) \leq (\bar{\lambda}_{fog} + \varepsilon) \log r \tag{39}$$

and

$$\log^{[2]} \nu_{fog}(r) \geq (\bar{\rho}_{fog} - \varepsilon) \log r \tag{40}$$

Now from (33) and (38), it follows for all sufficiently large values of r

$$\frac{\log^{[2]} \nu_{fog}(r)}{\log^{[2]} \nu_g(r)} \geq \frac{\bar{\lambda}_{fog} - \varepsilon}{\bar{\rho}_g + \varepsilon}$$

As $\varepsilon(> 0)$ is arbitrary, we obtain that

$$\liminf_{r\to\infty} \frac{\log^{[2]} \nu_{fog}(r)}{\log^{[2]} \nu_g(r)} \geq \frac{\bar{\lambda}_{fog}}{\bar{\rho}_g} \tag{41}$$

Again, combining (34) and (39) we get for a sequence of values of r tending to infinity

$$\frac{\log^{[2]} \nu_{fog}(r)}{\log^{[2]} \nu_g(r)} \leq \frac{\bar{\lambda}_{fog} + \varepsilon}{\bar{\lambda}_g - \varepsilon}$$

Since $\varepsilon(> 0)$ is arbitrary, it follows that

$$\liminf_{r\to\infty} \frac{\log^{[2]} \nu_{fog}(r)}{\log^{[2]} \nu_g(r)} \leq \frac{\bar{\lambda}_{fog}}{\bar{\lambda}_g} \tag{42}$$

Similarly, from (36) and (37) it follows for a sequence of values of r tending to infinity that

$$\frac{\log^{[2]} \nu_{fog}(r)}{\log^{[2]} \nu_g(r)} \leq \frac{\bar{\rho}_{fog} + \varepsilon}{\bar{\rho}_g - \varepsilon}$$

As $\varepsilon(> 0)$ is arbitrary, we obtain that

$$\liminf_{r\to\infty} \frac{\log^{[2]} \nu_{fog}(r)}{\log^{[2]} \nu_g(r)} \leq \frac{\bar{\rho}_{fog}}{\bar{\rho}_g} \tag{43}$$

Now combining (41), (42) and (43) we get that

$$\frac{\bar{\lambda}_{fog}}{\bar{\rho}_g} \leq \liminf_{r\to\infty} \frac{\log^{[2]} \nu_{fog}(r)}{\log^{[2]} \nu_g(r)} \leq \min\left\{\frac{\bar{\lambda}_{fog}}{\bar{\lambda}_g}, \frac{\bar{\rho}_{fog}}{\bar{\rho}_g}\right\} \tag{44}$$

Now from (35) and (38) we obtain for a sequence of values of r tending to infinity that

$$\frac{\log^{[2]} \nu_{fog}(r)}{\log^{[2]} \nu_g(r)} \geq \frac{\bar{\lambda}_{fog} - \varepsilon}{\bar{\lambda}_g + \varepsilon}$$

Choosing $\varepsilon \to 0$ we get that

$$\limsup_{r\to\infty}\frac{\log^{[2]}\nu_{fog}(r)}{\log^{[2]}\nu_g(r)}\geq\frac{\bar{\lambda}_{fog}}{\bar{\lambda}_g} \tag{45}$$

Again, from (34) and (37) it follows for all sufficiently large values of r

$$\frac{\log^{[2]}\nu_{fog}(r)}{\log^{[2]}\nu_g(r)}\leq\frac{\bar{\rho}_{fog}+\varepsilon}{\bar{\lambda}_g-\varepsilon}$$

As $\varepsilon(>0)$ is arbitrary, we obtain that

$$\limsup_{r\to\infty}\frac{\log^{[2]}\nu_{fog}(r)}{\log^{[2]}\nu_g(r)}\leq\frac{\bar{\rho}_{fog}}{\bar{\lambda}_g} \tag{46}$$

Similarly combining (33) and (40) we get for a sequence of values of r tending to infinity that

$$\frac{\log^{[2]}\nu_{fog}(r)}{\log^{[2]}\nu_g(r)}\geq\frac{\bar{\rho}_{fog}-\varepsilon}{\bar{\rho}_g+\varepsilon}$$

Since $\varepsilon(>0)$ is arbitrary, it follows that

$$\limsup_{r\to\infty}\frac{\log^{[2]}\nu_{fog}(r)}{\log^{[2]}\nu_g(r)}\geq\frac{\bar{\rho}_{fog}}{\bar{\rho}_g} \tag{47}$$

Therefore combining (45), (46) and (47) we get that

$$\max\left\{\frac{\bar{\lambda}_{fog}}{\bar{\lambda}_g},\frac{\bar{\rho}_{fog}}{\bar{\rho}_g}\right\}\leq\limsup_{r\to\infty}\frac{\log^{[2]}\nu_{fog}(r)}{\log^{[2]}\nu_g(r)}\leq\frac{\bar{\rho}_{fog}}{\bar{\lambda}_g} \tag{48}$$

Thus the theorem follows from (44) and (48).

Example 3: Let $= z$, $g=\exp^{[2]}$ z. Then it can be easily shown that the sign '$\leq$' in Theorem 3 cannot be replaced by '$<$'only

Remark 2: If we take $0<\bar{\lambda}_f\leq\bar{\rho}_f<\infty$ instead of $0<\bar{\lambda}_g\leq\bar{\rho}_g<\infty$ and the other conditions remain the same then also Theorem 3 holds with g replaced by f in the denominator as we see in the next theorem.

Theorem 4: Let f and g be two entire functions. Also let $0<\bar{\lambda}_{fog}\leq\bar{\rho}_{fog}<\infty$ and $0<\bar{\lambda}_f\leq\bar{\rho}_f<\infty$. Then

$$\frac{\bar{\lambda}_{fog}}{\bar{\rho}_f}\leq\liminf_{r\to\infty}\frac{\log^{[2]}\nu_{fog}(r)}{\log^{[2]}\nu_f(r)}\leq\min\left\{\frac{\bar{\lambda}_{fog}}{\bar{\lambda}_f},\frac{\bar{\rho}_{fog}}{\bar{\rho}_f}\right\}\leq\max\left\{\frac{\bar{\lambda}_{fog}}{\bar{\lambda}_f},\frac{\bar{\rho}_{fog}}{\bar{\rho}_f}\right\}$$

$$\leq\limsup_{r\to\infty}\frac{\log^{[2]}\nu_{fog}(r)}{\log^{[2]}\nu_f(r)}\leq\frac{\bar{\rho}_{fog}}{\bar{\lambda}_f}$$

Proof: From the definition of hyper order and hyper lower order of an entire function f we have for arbitrary positive ε and for all sufficiently large values of r that

$$\log^{[2]} \nu_f(r) \leq (\bar{\rho}_f + \varepsilon) \log r \tag{49}$$

and

$$\log^{[2]} \nu_f(r) \geq (\bar{\lambda}_f - \varepsilon) \log r \tag{50}$$

Also, for a sequence of values of r tending to infinity

$$\log^{[2]} \nu_f(r) \leq (\bar{\lambda}_f + \varepsilon) \log r \tag{51}$$

and

$$\log^{[2]} \nu_f(r) \geq (\bar{\rho}_f - \varepsilon) \log r \tag{52}$$

Again from the definition of hyper order and hyper lower order of the composite entire function fog, we have for arbitrary positive ε and for all sufficiently large values of r

$$\log^{[2]} \nu_{fog}(r) \leq (\bar{\rho}_{fog} + \varepsilon) \log r \tag{53}$$

and

$$\log^{[2]} \nu_{fog}(r) \geq (\bar{\lambda}_{fog} - \varepsilon) \log r \tag{54}$$

Again, for a sequence of values of r tending to infinity

$$\log^{[2]} \nu_{fog}(r) \leq (\bar{\lambda}_{fog} + \varepsilon) \log r \tag{55}$$

and

$$\log^{[2]} \nu_{fog}(r) \geq (\bar{\rho}_{fog} - \varepsilon) \log r \tag{56}$$

Now from (49) and (54) it follows for all sufficiently large values of r that

$$\frac{\log^{[2]} \nu_{fog}(r)}{\log^{[2]} \nu_f(r)} \geq \frac{\bar{\lambda}_{fog} - \varepsilon}{\bar{\rho}_f + \varepsilon}$$

As $\varepsilon(> 0)$ is arbitrary, we obtain that

$$\liminf_{r\to\infty} \frac{\log^{[2]} \nu_{fog}(r)}{\log^{[2]} \nu_f(r)} \geq \frac{\bar{\lambda}_{fog}}{\bar{\rho}_f} \tag{57}$$

Again, combining (50) and (55) we get for a sequence of values of r tending to infinity

$$\frac{\log^{[2]} \nu_{fog}(r)}{\log^{[2]} \nu_f(r)} \leq \frac{\bar{\lambda}_{fog} + \varepsilon}{\bar{\lambda}_f - \varepsilon}$$

Since $\varepsilon(> 0)$ is arbitrary, it follows that

$$\liminf_{r\to\infty}\frac{\log^{[2]}\nu_{fog}(r)}{\log^{[2]}\nu_f(r)}\le\frac{\bar{\lambda}_{fog}}{\bar{\lambda}_f} \tag{58}$$

Similarly, from (52) and (53) it follows for a sequence of values of r tending to infinity

$$\frac{\log^{[2]}\nu_{fog}(r)}{\log^{[2]}\nu_f(r)}\le\frac{\bar{\rho}_{fog}+\varepsilon}{\bar{\rho}_f-\varepsilon}$$

As $\varepsilon(>0)$ is arbitrary, we obtain that

$$\liminf_{r\to\infty}\frac{\log^{[2]}\nu_{fog}(r)}{\log^{[2]}\nu_f(r)}\le\frac{\bar{\rho}_{fog}}{\bar{\rho}_f} \tag{59}$$

Now combining (57), (58) and (59) we get that

$$\frac{\bar{\lambda}_{fog}}{\bar{\rho}_f}\le\liminf_{r\to\infty}\frac{\log^{[2]}\nu_{fog}(r)}{\log^{[2]}\nu_f(r)}\le\min\left\{\frac{\bar{\lambda}_{fog}}{\bar{\lambda}_f},\frac{\bar{\rho}_{fog}}{\bar{\rho}_f}\right\} \tag{60}$$

Now, from (51) and (54) we obtain for a sequence of values of r tending to infinity

$$\frac{\log^{[2]}\nu_{fog}(r)}{\log^{[2]}\nu_f(r)}\ge\frac{\bar{\lambda}_{fog}-\epsilon}{\bar{\lambda}_f+\varepsilon}$$

Choosing $\varepsilon\to 0$ we get that

$$\limsup_{r\to\infty}\frac{\log^{[2]}\nu_{fog}(r)}{\log^{[2]}\nu_f(r)}\ge\frac{\bar{\lambda}_{fog}}{\bar{\lambda}_f} \tag{61}$$

Again, from (50) and (53) it follows for all sufficiently large values of r

$$\frac{\log^{[2]}\nu_{fog}(r)}{\log^{[2]}\nu_f(r)}\le\frac{\bar{\rho}_{fog}+\varepsilon}{\bar{\lambda}_f-\varepsilon}$$

As $\varepsilon(>0)$ is arbitrary, we obtain that

$$\limsup_{r\to\infty}\frac{\log^{[2]}\nu_{fog}(r)}{\log^{[2]}\nu_f(r)}\le\frac{\bar{\rho}_{fog}}{\bar{\lambda}_f} \tag{62}$$

Similarly, combining (49) and (56) we get for a sequence of values of r tending to infinity

$$\frac{\log^{[2]}\nu_{fog}(r)}{\log^{[2]}\nu_f(r)}\ge\frac{\bar{\rho}_{fog}-\varepsilon}{\bar{\rho}_f+\varepsilon}$$

Since $\varepsilon(>0)$ is arbitrary, it follows that

$$\limsup_{r\to\infty}\frac{\log^{[2]}\nu_{fog}(r)}{\log^{[2]}\nu_f(r)}\ge\frac{\bar{\rho}_{fog}}{\bar{\rho}_f} \tag{63}$$

Therefore, combining (61), (62) and (63) we get

$$\max\left\{\frac{\bar{\lambda}_{fog}}{\bar{\lambda}_f},\frac{\bar{\rho}_{fog}}{\bar{\rho}_f}\right\}\le\limsup_{r\to\infty}\frac{\log^{[2]}\nu_{fog}(r)}{\log^{[2]}\nu_f(r)}\le\frac{\bar{\rho}_{fog}}{\bar{\lambda}_f}. \qquad (64)$$

Thus the theorem follows from (60) and (64).

Example 4: Considering $f=\exp^{[2]}z,\ g=z$ one can easily verify that the sign '$\le$' in Theorem 4 cannot be replaced by '$<$' only.

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