International Journal of Engineering & Scientific Research
Vol. 4 Issue 9, September 2016,
ISSN: 2347-6532 Impact Factor: 5.900
Journal Homepage: http://www.ijmra.us, Email: editorijmie@gmail.com
Double-Blind Peer Reviewed Refereed Open Access International Journal - Included in the International Serial Directories Indexed & Listed at: Ulrich's Periodicals Directory ©, U.S.A., Open J-Gage as well as in Cabell's Directories of Publishing Opportunities, U.S.A

ALLOCATING FEEDBACK STRATEGIES IN GENERAL MULTICHANNEL QUEUING PROBLEMS WITH BALKING AND RENEGING

Meenu Gupta*

Man Singh**

Deepak Gupta***

Abstract

The objective of this paper is to treat the solution of network of general queuing system having multi parallel server serial queues and single server non serial queues with balking and reneging considering the concepts of feedback. In **t**his paper we analyze steady – state analysis of a model of a general queuing system having M service channels in series being connected with N non-serial channels having reneging and balking phenomenon, where each of M service channels has identical multiple parallel channels and there is feedback to all the previous channels from the last multiple channels in series with certain probabiliies. Purpose of this model is to allow feedback to all available serial channels and to derive steady – state expressions for such queuing model . Here the queue discipline is random selection for service. Poisson arrivals and exponential holding times are followed. The customer becomes impatient in queue after sometime and may leave the system without getting service. The input process depends upon the queue size in serial and non-serial channels. The two models studied are as follows:

* **Asstt. Prof., Dept. of Mathematics, Dr. B. R.A. Govt. College,Kaithal-136027(Haryana)–India**

** **ExProf., CCS, Haryana Agricultural University , Hisar – India.**

*** **Prof. & Head ,Dept. of Mathematics, M.M.Engg.college, Mullana (Ambala)- India.**

Model-A: *There is feedback to all the previous channels from the last multiple channels in series with certain probabiliies. Waiting space is infinite.*

Model-B: *There is feedback to all the previous channels from the last multiple channels in series with certain probabiliies. Waiting space is finite.*

Keywords: Poisson stream, Reneging, Balking, Feedback, Steady-state, Parallel channels.

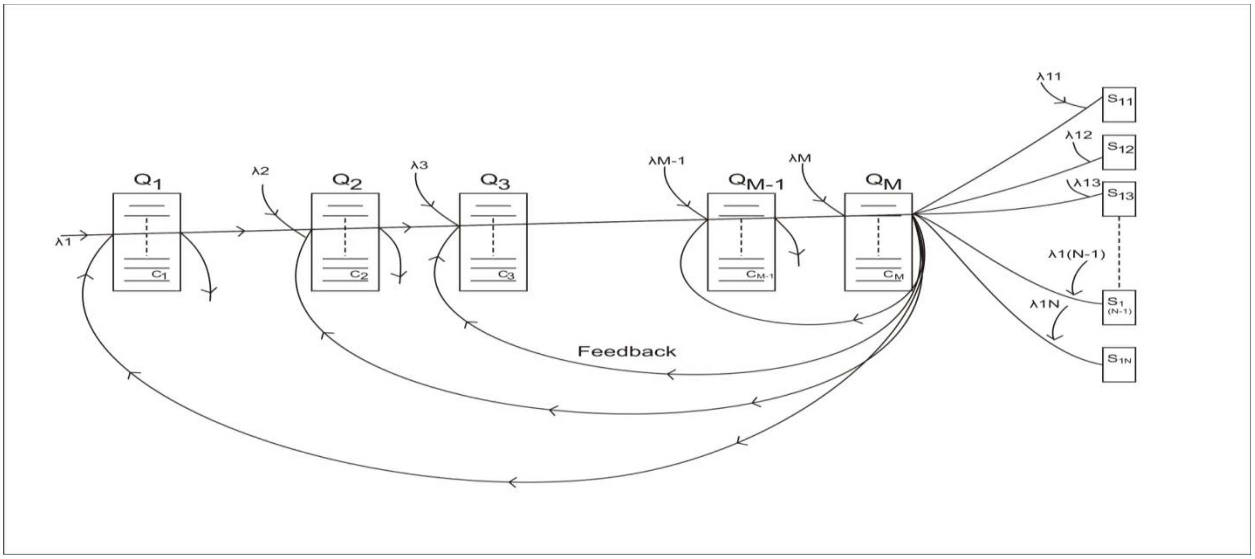
1. Introduction

At the present stage of development of queuing theory , queuing systems with parallel servers is one of the most extensively studied areas. The range of application of such queuing system is rather wide, for example, the modern information and computer systems, the design of modern computer networks (one of the basic principles is parallel information processing). Therefore new mathenmatical models of queuing systems are required. The problem of serial queues was studied by O' Brien (1954), Jackson (1954), and Hunt(1955) studied the problems of serial queues in the steady state with Poisson assumptions. In these studies it is assumed that the unit must go through each service channel without leaving the system. Barrer (1995) obtained the steady - state solution of a single channel queuing model having poisson input, exponential holding time, random selection where impatient customers leave the service facility after a wait of certain time. Finch (1959) studied simple queues with customers at random for service at a number of service stations in series where the arrival from outside was considered at the initial stage. Feedback is permitted either from the terminal server or from each server of the series to the queue waiting for service at that stage by imposing an upper limit on the number of customers in the system at any time. Maggu (1970) worked on certain types of queues in series. Singh (1984) studied the Steady-state behavior of serial queuing processes introducing the concept of reneging. Singh and Umed (1994) worked on the network of serial and non- serial queuing processes with impatient customers. Interrelationship between queuing models with balking and reneging was formalized by Gupta (1994). Singh and Ahuja (1996) derived results for the queuing model having multiple parallel channels in series with impatient customers. Here each serial server is a group of multiple parallel servers. Singh, Punam and Ashok (2008) obtained Steady-state solutions of serial and non-serial queuing processes with reneging and balking due to long queue and some urgent message and feedback. Meenu, Singh and Deepak (2012) formulated the steady-state solutions of multiple parallel channels in series connected to Non-serial multiple parallel channels both with balking &reneging. Also Satyabir and Singh (2012) derived the Steady-state Solution of Serial channels with Feedback and Balking connected with Non-Serial Queuing Processes with Reneging and Balking. Here we study the steady-state analysis of the present queuing system in the sense that M service channels in series are linked with N non-serial channels having reneging and balking phenomenon where each of M service channels has identical multiple parallel channels and there is feedback to all the previous channels from the last multiple parallel channels in series with certain probabilities. The input process is Poisson and the service time distribution is exponential. The service discipline follows SIRO-rule (Service in random order) instead of FIFO-rule (first in first out). The customer becomes impatient in queue after sometime and may leave the system without getting service. The input process depends upon the queue size in non-serial channels. Waiting space is infinite in Model –A and finite in Model-B and the results obtained are quoted in this present paper.

2. Model – A

2.1 Formulation of model

In this model there is feedback to all the previous channels from the last channel in series with certain probabiliies. The system consists of Q_i (i =1,2,3,….M) service phases in series and Q_{1j} non-serial service channels (j=1,2,3,….N) with respective servers S_i (i=1,2,3,……M) and S_{1j} (j=1,2,3,……N).Customers demanding different types of service arrive from outside the system in Poisson's stream with parameters λ_i(i =1,2,….M) at Q_i service phase and λ_{1j}(j=1,2,…..N) at Q_{1j} service phase respectively . But the fresh customers may decide not to enter the service channel Q_{1j} (j=1,2,….N) on seeing the long queue at Q_{1j}, then the Poisson input rate λ_{1j} would be $\frac{\lambda_{1j}}{m_j+1};\left(j=1,2,\ldots..N\right)$ where m_j is the queue size of Q_{1j}. Further, the impatient customers joining any service channel may leave the queue without getting service after a wait of certain time. The service time distributions for the servers S_i (i=1,2,3,……,M) and S_{1j} (j=1,2,3,…..,N) are mutually independent negative exponential distribution with service rates $\mu_i(i=1,2,3,\ldots\ldots..,M)$ and $\mu_{1j}(j=1,2,3,\ldots.,N)$ respectively. After the completion of service at any service phase in series, the customer either leaves the system with probability p_i or joins the next phase with probability q_i such that $p_i + q_i = 1$ (i= 1,2,3,…..M-1). After completion of service at Q_M , the customer either leaves the system with probability p_M or join any service phase Q_i (i =1,2,3,….M) back with respective probabilities s_i (i =1,2,3,….M-1) or join any of Q_{1j} (j=1,2,3,…….,N) with probabilities $\frac{q_{Mj}}{m_j+1}$ (j=1,2,3,….,N) such that $p_M + \sum_{i=1}^{M-1} s_i + \sum_{j=1}^{N} \frac{q_{Mj}}{m_j+1}$ =1.



2.2 Formulation of Equations

Define P(n_1 ,n_2 , ……….n_M ; m_1 ,m_2,m_3, …………m_N ;t) = Probability that at time ‘t’ , there are n_i (i=1,2,3,…….., M-1) customers (which may leave the system after being serviced by the ith server or join the next server) waiting before the servers S_i (i=1,2,3……..,M-1) ; n_M customers (may leave the system after being serviced by server S_M or join back any of the servers S_i (i=1,2,3……..,M-1) or join any of the queues Q_{1j}) waiting in the service channel Q_M ; m_j (j=1,2,3,…..,N) customers (which may balk or renege) waiting respectively before the servers S_{1j} (j=1,2,3…….N) . We define the operators $T_{i\square}$, $T_{\square i}$ and $T_{\square i,i+1\square}$ and $T'_{\square Mi}$ ($\tilde{n}$) to act upon the vector $\tilde{n} = (n_1, n_2, ..., n_M)$ and $T_{j\square}$ and $T_{\square j}$ and $T_{\square j,j+1\square}$ to act upon the vector $\tilde{m} = (m_1, m_2, ..., m_N)$ as follows:

$T_{i\square}(\tilde{n}) = (n_1, n_2, \ldots\ldots\ldots n_i - 1 \ldots n_M)$

$T_{\square i}(\tilde{n}) = (n_1, n_2, \ldots\ldots\ldots n_i + 1 \ldots n_M)$

$T_{\square i,i+1\square}(\tilde{n}) = (n_1, n_2, \ldots, n_i + 1, n_{i+1} - 1, \ldots, n_M)$ `

$T'_{\square Mi}(\tilde{n}) = (n_1, n_2, \ldots\ldots, n_i - 1, \ldots\ldots, n_M + 1)$

$T_{j\square}(\tilde{m}) = (m_1, m_2, \ldots\ldots m_j - 1 \ldots m_N)$

$T_{\square j}(\tilde{m}) = (m_1, m_2, \ldots\ldots m_j + 1 \ldots m_N)$

$T_{\square j,\, j+1\ \square}(\tilde{m}) = (m_1, m_2, \ldots\ldots, m_j + 1, m_{j+1} - 1 \ldots m_N)$

we write difference differential equations as:

$$\frac{dP(\tilde{n},\tilde{m};t)}{dt} = \left\{ \begin{array}{l} -\left[\sum_{i=1}^{M} \lambda_i + \sum_{j=1}^{N} \frac{\lambda_{1j}}{m_j + 1} + \sum_{i=1}^{M} \delta(n_i)\mu_{in_i} + \sum_{j=1}^{N} \delta(m_j)\{\mu_{1j} + R_{jm_j}\} \right] P(\tilde{n},\tilde{m};t) \\ + \sum_{i=1}^{M} \lambda_i P(T_{i\,\square}(\tilde{n}),\tilde{m};t)) + \sum_{j=1}^{N} \frac{\lambda_{1j}}{m_j} P(\tilde{n}; T_{j\,\square}(\tilde{m});t) + \sum_{i=1}^{M-1} q_i \mu_{in_i+1} P(T_{\square\, i\ ,\, i+1\ \square}(\tilde{n}),\tilde{m};t) \\ + \sum_{i=1}^{M} p_i \mu_{in_i+1} P(T_{\square i}(\tilde{n}),\tilde{m};t) + \sum_{i=1}^{M-1} s_i \mu_{M\ n_M+1} P(T'_{\ \square Mi},\tilde{m};t) \\ + \sum_{j=1}^{N} \mu_{Mn_M+1} \frac{q_{Mj}}{m_j} P(n_1, n_2, \ldots.. n_M + 1, T_{j\square}(\tilde{m});t) + \sum_{j=1}^{N} (\mu_{1j} + R_{jm_j+1}) P(\tilde{n}; T_{\square\ j}(\tilde{m});t) \end{array} \right\} \ldots\ldots.(1)$$

for $n_i \geq 0$, $mj \geq 0$; (i = 1,2,3,……..,M) ; (j= 1,2,3……..,N)

2.3 *Steady-State Equations*

We write the following steady-state equations of the queuing model by equating the time derivative to zero in the equation (1)

$$\left[\left(\sum_{i=1}^{M} \lambda_i + \sum_{j=1}^{N} \frac{\lambda_{1j}}{m_j + 1} + \sum_{i=1}^{M} \delta(n_i)\mu_{in_i} + \sum_{j=1}^{N} \delta(m_j)\{\mu_{1j} + R_{jm_j}\} \right) P(\tilde{n},\tilde{m}) \right]$$

$$= \left\{ \begin{array}{l} \sum_{i=1}^{M} \lambda_i P(T_{i\,\square}(\tilde{n}),\tilde{m})) + \sum_{j=1}^{N} \frac{\lambda_{1j}}{m_j} P(\tilde{n}; T_{j\,\square}(\tilde{m})) + \sum_{i=1}^{M-1} q_i \mu_{in_i+1} P(T_{\square\, i\ ,\, i+1\ \square}(\tilde{n}),\tilde{m}) \\ + \sum_{i=1}^{M} p_i \mu_{in_i+1} P(T_{\square i}(\tilde{n}),\tilde{m}) + \sum_{i=1}^{M-1} s_i \mu_{M\ n_M+1} P(T'_{\ \square Mi},\tilde{m}) \\ + \sum_{j=1}^{N} \mu_{Mn_M+1} \frac{q_{Mj}}{m_j} P(n_1, n_2, \ldots.. n_M + 1, T_{j\square}(\tilde{m})) + \sum_{j=1}^{N} (\mu_{1j} + R_{jm_j+1}) P(\tilde{n}; T_{\square\ j}(\tilde{m})) \end{array} \right\} \quad \ldots\ldots.(2)$$

for $n_i \geq 0$, $mj \geq 0$; (i = 1,2,3,……..,M) ; (j= 1,2,3……..,N)

Two cases arise depending upon the number of customers n_i and the number of channels c_i at Q_i service channel (i= 1,2,3………,M).

<u>Case (1) For $n_i < c_i$</u>

When the number of customers n_i before Q_i phase is less than the number of identical service channels c_i (i.e. $n_i < c_i$; i = 1, 2, 3,…, M) , then the service is immediately available to the customers on arrival and $\mu_{in_i} = n_i \mu_i$

2.4 Steady State Equations for $n_i < c_i$

For $n_i < c_i$, the resulting equations (2) reduce to as under:

$$\left[\sum_{i=1}^{M}\lambda_i + \sum_{j=1}^{N}\frac{\lambda_{1j}}{m_j+1} + \sum_{i=1}^{M}\delta(n_i)(n_i\mu_i) + \sum_{j=1}^{N}\delta(m_j)\{(\mu_{1j}) + R_{jm_j}\}\right]P(\tilde{n},\tilde{m})$$

$$= \left\{\begin{array}{l} \sum_{i=1}^{M}\lambda_i P(T_{i\cdot}(\tilde{n}),\tilde{m})) + \sum_{j=1}^{N}\frac{\lambda_{1j}}{m_j}P(\tilde{n};T_{j\cdot}(\tilde{m})) + \sum_{i=1}^{M-1} q_i\mu_i(n_i+1)P(T_{\cdot i,i+1\cdot}(\tilde{n}),\tilde{m}) \\ +\sum_{i=1}^{M} p_i\mu_i(n_i+1)P(T_{\cdot i}(\tilde{n}),\tilde{m}) + \sum_{j=1}^{N}\mu_M(n_M+1)\frac{q_{Mj}}{m_j}P(n_1,n_2,.....n_M+1,T_{j\cdot}(\tilde{m})) \\ +\sum_{i=1}^{M-1} s_i\mu_M(n_M+1)P(T'_{Mi}(\tilde{m})) + \sum_{j=1}^{N}(\mu_{1j}+R_{j(m_j+1)})P(\tilde{n};T_{\cdot j}(\tilde{m})) \end{array}\right\} .....(3)$$

for $n_i \geq 0$, $n_i < c_i$, $mj \geq 0$; (i = 1,2,3,……..,M) ; (j= 1,2,3……..,N) .

2.5 Steady State Solutions for $n_i < c_i$

The solutions of the steady state equations (3) can be verified to be:

$$P(\tilde{n},\tilde{m}) = \left\{\begin{array}{l} P(\tilde{0},\tilde{0})\left[\left(\frac{1}{\lfloor\underline{n_1}}\right)(\rho_1)^{n_1}\right]\left[\left(\frac{1}{\lfloor\underline{n_2}}\right)(\rho_2)^{n_2}\right]\left[\left(\frac{1}{\lfloor\underline{n_3}}\right)(\rho_3)^{n_3}\right]........\left[\left(\frac{1}{\lfloor\underline{n_M}}\right)(\rho_M)^{n_M}\right] \\ \left[\left(\frac{1}{\lfloor\underline{m_1}}\right)\left(\frac{(\lambda_{11}+\mu_M q_{M1}\rho_M)^{m_1}}{\prod_{j=1}^{m_1}(\mu_{11}+R_{1j})}\right)\right]\left[\left(\frac{1}{\lfloor\underline{m_2}}\right)\left(\frac{(\lambda_{12}+\mu_M q_{M2}\rho_M)^{m_2}}{\prod_{j=1}^{m_2}(\mu_{12}+R_{2j})}\right)\right]........ \\ ....\left[\left(\frac{1}{\lfloor\underline{m_N}}\right)\left(\frac{(\lambda_{1N}+\mu_M q_{MN}\rho_M)^{m_N}}{\prod_{j=1}^{m_N}(\mu_{1N}+R_{Nj})}\right)\right] . \end{array}\right\} ......(4)$$

for $n_i \geq 0$, $n_i < c_i$, $mj \geq 0$; (i = 1,2,3,……..,M) ; (j= 1,2,3……..,N) . Where

$$\mu_1\rho_1 = \lambda_1 + \mu_M s_1 \rho_M$$
$$\mu_2\rho_2 = \lambda_2 + \lambda_1 q_1 + (s_2 + s_1 q_1)\mu_M \rho_M$$
$$\mu_3\rho_3 = \lambda_3 + \lambda_2 q_2 + \lambda_1 q_1 q_2 + (s_3 + s_2 q_2 + s_1 q_1 q_2)\mu_M \rho_M$$
$$\begin{aligned}\mu_4\rho_4 = {} & \lambda_4 + \lambda_3 q_3 + \lambda_2 q_2 q_3 + \lambda_1 q_1 q_2 q_3 + (s_4 + s_3 q_3 + s_2 q_2 q_3 \\ & + s_1 q_1 q_2 q_3)\mu_M \rho_M\end{aligned}$$

……………………………………………………………………………………

……………………………………………………………………………………

……………………………………………………………………………………

$$\begin{aligned}\mu_{M-1}\rho_{M-1} = {} & [\lambda_{M-1} + \lambda_{M-2} q_{M-2} + \lambda_{M-3} q_{M-3} q_{M-2} + \ldots\ldots\ldots \\ & \ldots. + \lambda_3 q_{M-2} q_{M-3} q_{M-4} \ldots\ldots. q_4 q_3 \\ & + \lambda_2 q_{M-2} q_{M-3} q_{M-4} \ldots\ldots. q_4 q_3 q_2 \\ & + \lambda_1 q_{M-2} q_{M-3} q_{M-4} \ldots\ldots. q_4 q_3 q_2 q_1] \\ & + [s_{M-1} + s_{M-2} q_{M-2} + s_2 q_{M-2} \ldots\ldots q_3 q_2 + s_1 q_{M-2} \ldots\ldots. q_2 q_1]\mu_M \rho_M \\ \mu_M \rho_M = {} & \lambda_M + \mu_{M-1} q_{M-1} \rho_{M-1}\end{aligned}$$

ρ_M is determined independent of any other ρ_i with the help of last two equations.

Case (2) For $n_i \geq c_i$

When the number of customers before Q_i service phase is more than or equal to the number of identical service channels c_i (i.e. $n_i \geq c_i$) then $\mu_{in_i} = c_i \mu_i$ and $\delta(n_i) = 1$.

2.6 Steady State Equations for $n_i \geq c_i$ Then the resulting equations (2) will reduce to as :

$$\left\{\left(\sum_{i=1}^{M}\lambda_i + \sum_{j=1}^{N}\frac{\lambda_{1j}}{m_j+1} + \sum_{i=1}^{M}(c_i\mu_i) + \sum_{j=1}^{N}\delta(m_j)\{(\mu_{1j}) + R_{jm_j}\}\right)P(\tilde{n},\tilde{m})\right\}$$
$$= \left\{\begin{aligned} & \sum_{i=1}^{M}\lambda_i P(T_{i\,\square}(\tilde{n}),\tilde{m})) + \sum_{j=1}^{N}\frac{\lambda_{1j}}{m_j}P(\tilde{n};T_{j\,\square}(\tilde{m})) + \sum_{i=1}^{M-1} q_i\mu_i c_i P(T_{\square\, i,\, i+1\,\square}(\tilde{n}),\tilde{m}) \\ & + \sum_{i=1}^{M} p_i\mu_i c_i P(T_{\square\, i}(\tilde{n}),\tilde{m}) + \sum_{j=1}^{N}\mu_M c_M \frac{q_{Mj}}{m_j} P(n_1, n_2, \ldots.. n_M + 1, T_{j\,\square}(\tilde{m})) \\ & + \sum_{i=1}^{M-1} s_i \mu_M c_M P(T'_{\square Mi}, (\tilde{m})) + \sum_{j=1}^{N}(\mu_{1j} + R_{j(m_j+1)})P(\tilde{n};T_{\square\, j}(\tilde{m}))\end{aligned}\right\} \ldots..(5)$$

for $n_i \geq c_i$, mj $\geq$ 0 ; (i = 1,2,3,……..,M) ; (j= 1,2,3……..,N) .

2.7 Steady State Solutions for $n_i \geq c_i$

The solutions of the steady state equations (5) can be verified to be

$$P(\tilde{n},\tilde{m}) = \left\{ \begin{array}{l} P(\tilde{0},\tilde{0}) \left[\prod_{i=1}^{M} (\rho_i^{'})^{n_i} \right] \left[\dfrac{(\lambda_{11} + \mu_M \rho_M^{'} q_{M1})^{m_1}}{\underline{|m_1} \prod_{j=1}^{m_1} (\mu_{11} + R_{1j})} \right] \\ \left[\left[\dfrac{(\lambda_{12} + \mu_M \rho_M^{'} q_{M2})^{m_2}}{\underline{|m_2} \prod_{j=1}^{m_2} (\mu_{12} + R_{2j})} \right] \cdots\cdots\cdots \left[\dfrac{(\lambda_{1N} + \mu_M \rho_M^{'} q_{MN})^{m_N}}{\underline{|m_N} \prod_{j=1}^{m_N} (\mu_{1N} + R_{Nj})} \right] \right]. \end{array} \right\} \text{........(6)}$$

for $n_i \geq c_i$, mj ≥ 0 ; (i = 1,2,3,……..,M) ; (j= 1,2,3……..,N) .

Where

$$c_1 \mu_1 \rho_1^{'} = \lambda_1 + c_M \mu_M s_1 \rho_M^{'}$$

$$c_2 \mu_2 \rho_2^{'} = (\lambda_2 + \lambda_1 q_1) + (s_2 + s_1 q_1) c_M \mu_M \rho_M^{'}$$

$$c_3 \mu_3 \rho_3^{'} = (\lambda_3 + \lambda_2 q_2 + \lambda_1 q_1 q_2) + (s_3 + s_2 q_2 + s_1 q_1 q_2) c_M \mu_M \rho_M^{'}$$

$$\begin{aligned} c_4 \mu_4 \rho_4^{'} = {} & (\lambda_4 + \lambda_3 q_3 + \lambda_2 q_3 q_2 + \lambda_1 q_3 q_2 q_1) + (s_4 + s_3 q_3 + s_2 q_3 q_2 \\ & + s_1 q_3 q_2 q_1) c_M \mu_M \rho_M^{'} \end{aligned}$$

……………………………………………………………………………………………

……………………………………………………………………………………………

……………………………………………………………………………………………

$$\begin{aligned} c_{M-1} \mu_{M-1} \rho_{M-1}^{'} = {} & [\lambda_{M-1} + \lambda_{M-2} q_{M-2} + \lambda_{M-3} q_{M-3} q_{M-2} + \lambda_{M-4} q_{M-4} q_{M-3} q_{M-2} \\ & + \ldots\ldots\ldots\ldots\ldots\ldots + \lambda_3 q_{M-2} q_{M-3} q_{M-4} \ldots\ldots q_5 q_4 q_3 \\ & + \lambda_2 q_{M-2} q_{M-3} q_{M-4} \ldots\ldots\ldots\ldots\ldots\ldots q_4 q_3 q_2 \\ & + \lambda_1 q_{M-2} q_{M-3} q_{M-4} \ldots\ldots\ldots\ldots\ldots\ldots q_3 q_2 q_1] \\ & + [s_{M-1} + s_{M-2} q_{M-2} + s_{M-3} q_{M-3} q_{M-2} + s_{M-4} q_{M-4} q_{M-3} q_{M-2} \\ & + \ldots\ldots\ldots\ldots + s_2 q_{M-2} . q_{M-3} q_{M-4} \ldots\ldots\ldots\ldots q_3 q_2 \\ & + s_1 q_{M-2} . q_{M-3} q_{M-4} \ldots\ldots\ldots\ldots\ldots\ldots q_2 q_1] c_M \mu_M \rho_M^{'} \end{aligned}$$

$$c_M \mu_M \rho_M^{'} = \lambda_M + c_{M-1} \mu_{M-1} q_{M-1} \rho_{M-1}^{'}$$

We determine $\rho_M^{'}$ and $\rho_{M-1}^{'}$ with the help of last two equations and so each $\rho_i^{'}$ (i=1,2,3……,M-1,M) would be obtained from the above equations.

We obtain $P(\tilde{0},\tilde{0})$ from (4) and (6) by the normalizing condition $\sum_{\tilde{m}=\tilde{0}}^{\infty}\sum_{\tilde{n}=\tilde{0}}^{\infty}P(\tilde{n},\tilde{m})=1$ and with the restriction that the traffic intensity of each service channel of the system is less than unity. Thus $P(\tilde{n},\tilde{m})$ is completely determined.

3. *Model –B*

Here we assume that if at any instant there are K customers in the system $\sum_{i=1}^{M}n_i+\sum_{l=1}^{N}m_j=K$, then the customers arriving at that instant will not be allowed to join the system and is considered as forced balking and lost for the system. Define $P(\tilde{n},\tilde{m};t)$ and the operators $T_{i\square}$, $T_{\square i}$ and $T_{\square i,\,i+1\,\square}$ and $T^{`}_{\square Mi}(\tilde{n})$ to act upon the vector $\tilde{n}=(n_1,n_2,.............,n_M)$ and $T_{j\square}$ and $T_{\square j}$ and $T_{\square j,\,j+1\,\square}$ to act upon the vector $\tilde{m}=(m_1,m_2,.............,m_N)$ as in model A.

The following difference -differential equations hold :

$$\frac{dP(\tilde{n},\tilde{m};t)}{dt}=\left\{\begin{array}{l}
-\left[\sum_{i=1}^{M}\lambda_i+\sum_{j=1}^{N}\frac{\lambda_{1j}}{m_j+1}+\sum_{i=1}^{M}\delta(n_i)\mu_{in_i}+\sum_{j=1}^{N}\delta(m_j)\{\mu_{1j}+R_{jm_j}\}\right]P(\tilde{n},\tilde{m};t)\\
\qquad+\sum_{i=1}^{M}\lambda_i P(T_{i\,\square}(\tilde{n}),\tilde{m};t))+\sum_{j=1}^{N}\frac{\lambda_{1j}}{m_j}P(\tilde{n};T_{j\,\square}(\tilde{m});t)\\
\qquad+\sum_{i=1}^{M-1}q_i\mu_{in_i+1}P(T_{\square\, i\;,\,i+1\;\square}(\tilde{n}),\tilde{m};t)+\sum_{i=1}^{M}p_i\mu_{in_i+1}P(T_{\square i}(\tilde{n}),\tilde{m};t)\\
\qquad+\sum_{i=1}^{M-1}s_i\mu_{M\;n_M+1}P(T^{'}_{\square Mi}(\tilde{n}),\tilde{m};t)\\
\qquad+\sum_{j=1}^{N}\mu_{Mn_M+1}\frac{q_{Mj}}{m_j}P(n_1,n_2,.....n_M+1,T_{j\square}(\tilde{m});t)\\
\qquad+\sum_{j=1}^{N}(\mu_{1j}+R_{jm_j+1})P(\tilde{n};T_{\square\;j}(\tilde{m});t)
\end{array}\right\}.......(7)$$

for $n_i\geq 0$, $mj\geq 0$; (i = 1,2,3,……..,M) ; (j= 1,2,3……..,N); $\sum_{i=1}^{M}n_i+\sum_{l=1}^{N}m_j<K$

And

$$\frac{d}{dt}P(\tilde{n},\tilde{m};t)=\left\{\begin{array}{l} -\left[\sum_{i=1}^{M}\delta(n_i)\left(\mu_{in_i}\right)+\sum_{j=1}^{N}\delta(m_j)\left(\mu_{ij}+R_{jm_j}\right)\right]P(\tilde{n},\tilde{m};t)+ \\ +\sum_{i=1}^{M-1} q_i\mu_{in_i+1}P(T_{\square\, i\,,\,i+1\,\square}(\tilde{n}),\tilde{m};t)+\sum_{i=1}^{M-1}s_i\mu_{M\;n_M+1}P(T'_{\square Mi}(\tilde{n}),\tilde{m};t) \\ +\sum_{j=1}^{N}\mu_{Mn_M+1}\frac{q_{Mj}}{m_j}P(n_1,n_2,\ldots,n_M+1,T_{j\,\square}(\tilde{m});t) \\ +\sum_{i=1}^{M}\lambda_i P(T_{i\,\square}(\tilde{n}),\tilde{m};t)+\sum_{j=1}^{N}\frac{\lambda_{1j}}{m_j}P(\tilde{n},T_{j\,\square}(\tilde{m});t) \end{array}\right\} \text{..........(8)}$$

for $n_i \geq 0$, $mj \geq 0$; (i = 1,2,3,…….,M) ; (j= 1,2,3…….,N); $\sum_{i=1}^{M}n_i+\sum_{l=1}^{N}m_j=K$

3.1 Steady-State Equations

We obtain the following steady state equations of the queuing model by equating the time derivative to zero in the equation (7) and (8).

$$\left[\left(\sum_{i=1}^{M}\lambda_i+\sum_{j=1}^{N}\frac{\lambda_{1j}}{m_j+1}+\sum_{i=1}^{M}\delta(n_i)(\mu_{in_i})+\sum_{j=1}^{N}\delta(m_j)\{(\mu_{1j})+R_{jm_j}\}\right)P(\tilde{n},\tilde{m})\right]$$

$$=\left[\begin{array}{l} \sum_{i=1}^{M}\lambda_i P(T_{i\,\square}(\tilde{n}),\tilde{m}))+\sum_{j=1}^{N}\frac{\lambda_{1j}}{m_j}P(\tilde{n};T_{j\,\square}(\tilde{m}))+\sum_{i=1}^{M-1}q_i\mu_{i(n_i+1)}P(T_{\square\, i\,,\,i+1\,\square}(\tilde{n}),\tilde{m}) \\ +\sum_{i=1}^{M-1}s_i\mu_{M(n_M+1)}P(T'_{\square Mi}(\tilde{n}),\tilde{m})+\sum_{i=1}^{M}p_i\mu_{i(n_i+1)}P(T_{\square\, i}(\tilde{n}),\tilde{m})+ \\ \sum_{j=1}^{N}\mu_{M(n_M+1)}\frac{q_{Mj}}{m_j}P(n_1,n_2,.....n_M+1,T_{j\,\square}(\tilde{m}))+\sum_{j=1}^{N}(\mu_{1j}+R_{j(m_j+1)})P(\tilde{n};T_{\square\, j}(\tilde{m})) \end{array}\right] \text{.....(9)}$$

for $n_i \geq 0$, $mj \geq 0$; (i = 1,2,3,…….,M) ; (j= 1,2,3…….,N) ; $\sum_{i=1}^{M}n_i+\sum_{l=1}^{N}m_j<K$

And

$$\left[\left(\sum_{i=1}^{M}\delta(n_i)(\mu_{in_i})+\sum_{j=1}^{N}\delta(m_j)\{(\mu_{1j})+R_{jm_j}\}\right)P(\tilde{n},\tilde{m})\right]$$

$$=\left[\begin{array}{l} \sum_{i=1}^{M}\lambda_i P(T_{i\,\square}(\tilde{n},\tilde{m}))+\sum_{j=1}^{N}\frac{\lambda_{1j}}{m_j}P(\tilde{n};T_{j\,\square}(\tilde{m}))+\sum_{i=1}^{M-1}q_i\mu_{i(n_i+1)}P(T_{\square\, i\,,\,i+1\,\square}(\tilde{n}),\tilde{m}) \\ +\sum_{i=1}^{M-1}s_i\mu_{M(n_M+1)}P(T'_{\square Mi}(\tilde{n}),\tilde{m})+\sum_{j=1}^{N}\mu_{M(n_M+1)}\frac{q_{Mj}}{m_j}P(n_1,n_2,.....n_M+1,T_{j\,\square}(\tilde{m})) \end{array}\right] \text{......(10)}$$

for $n_i \geq 0$, mj ≥ 0 ; (i = 1,2,3,……..,M) ; (j= 1,2,3……..,N) ; $\sum_{i=1}^{M} n_i + \sum_{l=1}^{N} m_j = K$

Two cases arise depending upon the number of customers n_i and number of channels c_i at Q_i phase (i= 1,2,3………,M).

<u>Case (1) For $n_i < c_i$</u>

When the number of customers n_i before Q_i phase is less than the number of identical service channels c_i (i.e. $n_i < c_i$; i = 1, 2, 3,…, M) , then the service is immediately available to the customers on arrival and $\mu_{in_i} = n_i \mu_i$.

3.2 Steady State Equations for $n_i < c_i$

For $n_i < c_i$, the resulting equations (9) and (10) reduce to as under:

$$\left[\left(\sum_{i=1}^{M}\lambda_i + \sum_{j=1}^{N}\frac{\lambda_{1j}}{m_j+1} + \sum_{i=1}^{M}\delta(n_i)(n_i\mu_i) + \sum_{j=1}^{N}\delta(m_j)\{(\mu_{1j}) + R_{jm_j}\}\right)P(\tilde{n},\tilde{m})\right]$$

$$= \begin{bmatrix} \sum_{i=1}^{M}\lambda_i P(T_{i\,\cdot}(\tilde{n}),\tilde{m})) + \sum_{j=1}^{N}\frac{\lambda_{1j}}{m_j}P(\tilde{n};T_{j\,\cdot}(\tilde{m})) + \sum_{i=1}^{M-1}q_i\mu_i\,(n_i+1)P(T_{\cdot i,\, i+1\,\cdot}(\tilde{n}),\tilde{m}) \\ +\sum_{i=1}^{M-1}s_i\mu_M\,(n_M+1)P(T'_{\cdot Mi}(\tilde{n}),\tilde{m}) + \sum_{i=1}^{M}p_i\mu_i(n_i+1)P(T_{\cdot i}(\tilde{n}),\tilde{m}) \\ +\sum_{j=1}^{N}\mu_M\,(n_M+1)\frac{q_{Mj}}{m_j}P(n_1,n_2,.....n_M+1,T_{j\,\cdot}(\tilde{m})) + \sum_{j=1}^{N}(\mu_{1j}+R_{j(m_j+1)})P(\tilde{n};T_{\cdot j}(\tilde{m})) \end{bmatrix} \text{...(11)}$$

for $n_i \geq 0$, $n_i < c_i$, mj ≥ 0 ; (i = 1,2,3,……..,M) ; (j= 1,2,3……..,N) . $\sum_{i=1}^{M} n_i + \sum_{l=1}^{N} m_j < K$

And

$$\left[\left(\sum_{i=1}^{M}\delta(n_i)(\mu_i n_i) + \sum_{j=1}^{N}\delta(m_j)\{(\mu_{1j}) + R_{jm_j}\}\right)P(\tilde{n},\tilde{m})\right]$$

$$= \begin{bmatrix} \sum_{i=1}^{M}\lambda_i P(T_{i\,\cdot}(\tilde{n}),\tilde{m}) + \sum_{j=1}^{N}\frac{\lambda_{1j}}{m_j}P(\tilde{n};T_{j\,\cdot}(\tilde{m})) + \sum_{i=1}^{M-1}q_i\mu_i(n_i+1)P(T_{\cdot i\,,i+1\,\cdot}(\tilde{n}),\tilde{m}) \\ +\sum_{i=1}^{M-1}s_i\mu_M\,(n_M+1)P(T'_{\cdot Mi}(\tilde{n}),\tilde{m}) + + \sum_{j=1}^{N}\mu_M\,(n_M+1)\frac{q_{Mj}}{m_j}P(n_1,n_2,.....n_M+1,T_{j\,\cdot}(\tilde{m})) \end{bmatrix} \text{......(12)}$$

for $n_i \geq 0$, $n_i < c_i$, $mj \geq 0$; (i = 1,2,3,……..,M) ; (j= 1,2,3……..,N) . $\sum_{i=1}^{M} n_i + \sum_{l=1}^{N} m_j = K$

3.3 Steady State Solutions for $n_i < c_i$

The solutions of the steady state equations (11) and (12) can be verified to be:

$$P(\tilde{n},\tilde{m}) = \left[\begin{array}{l} P(\tilde{0},\tilde{0})\left[\left(\frac{1}{\lfloor n_1}\right)(\rho_1)^{n_1}\right] \left[\left(\frac{1}{\lfloor n_2}\right)(\rho_2)^{n_2}\right] \left[\left(\frac{1}{\lfloor n_3}\right)(\rho_3)^{n_3}\right] \\ \ldots\ldots\left[\left(\frac{1}{\lfloor n_M}\right)(\rho_M)^{n_M}\right] \left[\left(\frac{1}{\lfloor m_1}\right)\left(\frac{(\lambda_{11} + \mu_M q_{M1}\rho_M)^{m_1}}{\prod_{j=1}^{m_1}(\mu_{11} + R_{1j})}\right)\right] \\ \left[\left[\left(\frac{1}{\lfloor m_2}\right)\left(\frac{(\lambda_{12} + \mu_M q_{M2}\rho_M)^{m_2}}{\prod_{j=1}^{m_2}(\mu_{12} + R_{2j})}\right)\right] \ldots\ldots\ldots \left[\left(\frac{1}{\lfloor m_N}\right)\left(\frac{(\lambda_{1N} + \mu_M q_{MN}\rho_M)^{m_N}}{\prod_{j=1}^{m_N}(\mu_{1N} + R_{Nj})}\right)\right]\right] \end{array} \right] \ldots\ldots(13)$$

for $n_i \geq 0$, $n_i < c_i$, $mj \geq 0$; (i = 1,2,3,……..,M) ; (j= 1,2,3……..,N) . $\sum_{i=1}^{M} n_i + \sum_{l=1}^{N} m_j \leq K$

where $\rho_1, \rho_2, \rho_3, \rho_4 ...., \rho_M$. are defined in model A.

<u>Case(2) For $n_i \geq c_i$</u>

When the number of customers before Q_i service phase is more than or equal to the number of identical service channels c_i (i.e. $n_i \geq c_i$) then $\mu_{in_i} = c_i \mu_i$ and $\delta(n_i) = 1$.

3.4 Steady State Equations for $n_i \geq c_i$

The resulting equations (9) and (10) will reduce to as under:

$$\left[\left(\sum_{i=1}^{M}\lambda_i+\sum_{j=1}^{N}\frac{\lambda_{1j}}{m_j+1}+\sum_{i=1}^{M}c_i\mu_i+\sum_{j=1}^{N}\delta(m_j)\{(\mu_{1j})+R_{jm_j}\}\right)P(\tilde{n},\tilde{m})\right]$$

$$=\left[\begin{array}{l}\sum_{i=1}^{M}\lambda_i P(T_{i\cdot}(\tilde{n}),\tilde{m}))+\sum_{j=1}^{N}\frac{\lambda_{1j}}{m_j}P(\tilde{n};T_{j\cdot}(\tilde{m}))+\sum_{i=1}^{M-1}q_i\mu_i\;c_iP(T_{\cdot i,\,i+1\cdot}(\tilde{n}),\tilde{m})\\+\sum_{i=1}^{M-1}s_i\mu_M c_M P(T'_{\cdot Mi}(\tilde{n}),\tilde{m};t)+\sum_{i=1}^{M}p_i\mu_i c_i P(T_{\cdot i}(\tilde{n}),\tilde{m})\\+\sum_{j=1}^{N}\mu_M c_M\frac{q_{Mj}}{m_j}P(n_1,n_2,.....n_M+1,T_{j\cdot}(\tilde{m}))+\sum_{j=1}^{N}(\mu_{1j}+R_{j(m_j+1)})P(\tilde{n};T_{\cdot j}(\tilde{m}))\end{array}\right]....(14)$$

for $n_i \geq 0$, $n_i \geq c_i$, $m_j \geq 0$; (i = 1,2,3,……..,M) ; (j= 1,2,3……..,N) ; $\sum_{i=1}^{M}n_i+\sum_{l=1}^{N}m_j<K$

and

$$\left[\left(\sum_{i=1}^{M}(c_i\mu_i)+\sum_{j=1}^{N}\delta(m_j)\{(\mu_{1j})+R_{jm_j}\}\right)P(\tilde{n},\tilde{m})\right]$$

$$=\left[\begin{array}{l}\sum_{i=1}^{M}\lambda_i P(T_{i\cdot}(\tilde{n}),\tilde{m})+\sum_{j=1}^{N}\frac{\lambda_{1j}}{m_j}P(\tilde{n};T_{j\cdot}(\tilde{m}))+\sum_{i=1}^{M-1}q_i\mu_i c_i P(T_{\cdot i,\,i+1\cdot}(\tilde{n}),\tilde{m})\\+\sum_{i=1}^{M-1}s_i\mu_M c_M P(T'_{\cdot Mi}(\tilde{n}),\tilde{m})+\sum_{j=1}^{N}\mu_M c_M\frac{q_{Mj}}{m_j}P(n_1,n_2,.....n_M+1,T_{j\cdot}(\tilde{m})).\end{array}\right]...........(15)$$

for $n_i \geq 0$, $n_i \geq c_i$, $m_j \geq 0$; (i = 1,2,3,……..,M) ; (j= 1,2,3……..,N) ; $\sum_{i=1}^{M}n_i+\sum_{l=1}^{N}m_j=K$

3.5 Steady State Solutions for $n_i \geq c_i$

The solutions of the steady state equations (14) and (15) can be verified to be:

$$P(\tilde{n},\tilde{m})=\left[\begin{array}{l}P(\tilde{0},\tilde{0})\left[\prod_{i=1}^{M}(\rho_i')^{n_i}\right]\left[\left(\frac{1}{\lfloor\underline{m_1}}\right)\left(\frac{(\lambda_{11}+\mu_M q_{M1}\rho_M')^{m_1}}{\prod_{j=1}^{m_1}(\mu_{11}+R_{1j})}\right)\right]\\\left[\left(\frac{1}{\lfloor\underline{m_2}}\right)\left(\frac{(\lambda_{12}+\mu_M q_{M2}\rho_M')^{m_2}}{\prod_{j=1}^{m_2}(\mu_{12}+R_{2j})}\right)\right]..........\left[\left(\frac{1}{\lfloor\underline{m_N}}\right)\left(\frac{(\lambda_{1N}+\mu_M q_{MN}\rho_M')^{m_N}}{\prod_{j=1}^{m_N}(\mu_{1N}+R_{Nj})}\right)\right]\end{array}\right]......(16)$$

For $n_i \geq 0$, $n_i \geq c_i$, $mj \geq 0$; (i = 1,2,3,…….,M) ; (j= 1,2,3…….,N); $\sum_{i=1}^{M} n_i + \sum_{l=1}^{N} m_j \leq K$ .Where $\rho_1^{'}, \rho_2^{'}, \rho_3^{'}, \rho_4^{'}, \ldots\ldots\ldots\ldots\ldots\ldots, \rho_M^{'}$. have been explained in model A

We obtain $P(\tilde{0}, \tilde{0})$ from (13) and (16) by the normalizing condition $\sum_{\tilde{m}=\tilde{0}}^{\infty} \sum_{\tilde{n}=\tilde{0}}^{\infty} P(\tilde{n}, \tilde{m}) = 1$ and with the restrictions that $\sum_{i=1}^{M} n_i + \sum_{l=1}^{N} m_j \leq K$ the traffic intensity of each service channel of the system is less than unity. Thus $P\left(\tilde{n}, \tilde{m}\right)$ is completely determined.

4. Practical application

it is being realized to study the network of queuing process having reneging , balking and feedback on random service as such models are of common occurrence in our administrative set up where the people (customers) meet the administrators in connection with their problems and these administrators generally call the persons for hearing randomly.

The practical situations where such a model finds application are of common occurrence. For example, consider the administration of a particular state at the level of district head quarter consisting of Patwaris, Kanoongoes, Tehsildars, Sub-divisional magistrates, district commisioner etc. These officers correspond to the servers of serial channels. Education department, Health department, Irritation department etc. connected with the last server of serial queue correspond to non-serial channels. The people meet the officers of the district in connection with their problems. It is also a common practice that the officers call the people for hearing randomly .Further District commisioner may send the customers to different departments such as education, health, irrigation etc. if their problems are related to such departments. Concept of feedback to any of the previous serial server from the last serial server, considered in this model, will help customers to approach any previous serial server if it is so required.

References

1. **O'Brien, G.C.** (1954): The solutions of some queuing problems. ***J.Soc. Ind. Appl. Math. Vol. 2, 132-142.***
2. **Jackson, R.R.P.** (1954): Queuing systems with phase type service. ***Operations Research***

Quarterly. Vol. 5, No.2, 109-120.

3. **Barrer, D.Y.** (1955): A Waiting line problem characterized by impatient customers and indifferent clerk. ***Journal of Operations Research. Society of America. Vol.3, 360-367***
4. **Hunt, G.C**. (1956): Sequential arrays of waiting lines. ***Opps. Res. Vol. 4, 674-683.***
5. **Finch, P.D.(1959):** Cyclic Queues with Feedback. ***J. Roy Statist. Soc. B-21, 153-157.***
6. **Maggu, P.L. (**1970): On certain types of queues in series. ***Statistica Neerlandica. Vol. 24, 89-97***
7. **Singh, Man.**(1984): Steady-state behavior of serial queuing processes with impatient customers. ***Math. Operations forsch .U. Statist. Ser, statist. Vol. 15, No.2. 289-298.***
8. **Gupta, S.M.(1994):** Interrelationship between queuing models with Balking and Reneging and Machine repair problem with warm spares. ***Microelectron, Reliab., Vol. 34, No. 2, pp. 201-209.***
9. **Singh. Man and Singh, Umed (1994):** Network of Serial and non- serial queuing processes with impatient customers. ***JISSOR Vol. 15, pp. 1-4.***
10. **Singh Man and Ahuja Asha** (1996): The Steady State Solutions of Multiple Parallel Channels in Series With Impatient Customers. ***Intnl. J. Mgmt. Sys. Vol. 12, No. 1 .67-76.***
11. **Singh Man , Punam and Ashok Kumar** (2008): Steady-state solutions of serial and non-serial queuing processes with reneging and balking due to long queue and some urgent message and feedback . ***International Journal of Essential Science Vol. 2, No.2. 12-17.***
12. ***Punam; Singh, Man and Ashok Kumar (2011): The Steady –State solution of serial queuing processes with reneging due to long queue and urgent message. Applied Science Periodical Vol.XII, No. 1.***
13. **Singh, Satyabir and Singh, Man** (2012): The Steady-state Solution of Serial channels with Feedback and Balking connected with Non-Serial Queuing Processes with Reneging Balking. ***Global Journal of Theoretical and Applied Mathematical Sciences. Vol. 2, No. 1, pp 1-11.***
14. **Gupta, Meenu and Singh, Man and Gupta, Deepak** (2012): The steady-state solutions of multiple parallel channels in series and Non-serial multiple parallel channels both with

balking &reneging. ***International journal of computer application, Issue 2, Vol. 2, pp 119-126.***

15. **Gupta, Meenu and Singh, Man and Gupta, Deepak** (2014): Balking, Reneging and Feedback in general queuing model of multiple parallel channels with non- serial channels, ***International Journal of Advanced Scientific and Technical Research , Issue 4, Vol. 3, pp 483-503***