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Study of Universal Coefficient Theorem for Homology

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Abstract

This paper will give a brief introduction to homological algebra. Starting with various exact sequences, we will define tensor product and projective modules, which will lead to the object of interest: homology groups, a more computable alternative to homotopy groups in higher dimensions. Given a chain complex of free abelian groups Cn, is it possible to compute the homology groups Hn(C; G) of the associated chain complex of tensor product with G just in terms of G and Hn(C). The Universal Coefficient Theorem for Homology provides an algebraic formula that answers this question.

Introduction

In algebraic topology, we can distinguish various topological spaces using singular homology. Nonetheless we may want to calculate homology of arbitrary coefficients, so we need a theorem which will establish the relationship between homology of arbitrary coefficients and homology with Z coefficients. In this article we will give the necessary algebra background as well as we will define Tor and prove the Universal Coefficient Theorem for Homology.

Background in Algebra

1.Exact Sequences

Definition 1. A pair of homomorphisms $A \xrightarrow{f} B \xrightarrow{g} C$ is exact at B if im (f) =ker(g). A sequence $\cdot \cdot \cdot \to A_{i-1} \to A_i \to A_{i+1} \to \cdot \cdot \cdot$ is exact if it is exact at every Ai that is between two homomorphisms.

Proposition 2. A sequence $0 \to A \xrightarrow{f} B$ is exact if and only if f is injective. On the other hand, a sequence $B \xrightarrow{g} C \to 0$ is exact if and only if g is surjective.

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Proof. Exactness at A implies that ker f is equal to the image of the homomorphism $0 \to A$, which is zero. This is equivalent to the injectivity of homomorphism f. Similarly, the kernel of zero homomorphism $C \to 0$ is C, and g(B) = C if and only if g is surjective.

Corollary 3. A sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is exact if and only if f is injective, g is surjective, and im f = ker g. We say B is an extension of C by A. This exact sequence is called a short exact sequence.

Definition 4. Let $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$be a short exact sequence of R-modules. The sequence is split if $B = A \oplus C$ up to isomorphism. A map $s : C \to B$ is called a section of g if $g \circ s$ =id. If s is also a homomorphism, then it is a splitting homomorphism.

Splitting is equivalent to either of the following statements:

(a) There is a homomorphism $p : B \to A$ such that $p \circ f = 1 : A \to A$.

(b) There is a homomorphism $s : C \to B$ such that $g \circ s = 1 : C \to C$.

2. Tensor Product of Modules

Definition 2.1. For a ring R, let M be a right module, and N be a left module. The tensor product $M \otimes N$ over R is the abelian group M X N quotient by

(m1 + m2, n) ~ (m1, n) + (m2, n)

(m, n1 + n2) ~ (m, n1) + (m, n2)

(mr, n) ~ (m, rn)

for m, m1, m2 ∈ M, n, n1, n2 ∈ N and r ∈ R.

Theorem 2.2. Let L, M, N be right modules, and D be a left module.

If $0 \to L \xrightarrow{\Psi} M \xrightarrow{\phi} N \to 0$ is exact, then the associated sequence of abelian groups

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$L \otimes_R D \xrightarrow{\psi\otimes 1} M \otimes_R D \xrightarrow{\phi\otimes 1} N \otimes_R D \to 0$ is exact.

3. Universal Coefficient Theorem

We defined the homology groups $H_n(x)$ of a topological space X as

$$H_n(X) = H_n(S(X)),$$

where *S(X)* is the singular complex of *X*. It is often convenient to modify this construction by allowing "coefficients" in an abelian group G:

$$H_n(X;G) = H_n(S(X) \otimes G)$$

(one says "coefficients" because a typical element of degree *n* in $S(X) \otimes G$ has the form $\sum s_i \otimes g_i$, where $s_i \in S_n(X)$ and $g_i \in G$). For example, "obstruction theory" deals with the problem of extending a continuous map defined on a "nice" subspace of *X* and involves cohomology with coefficients in a certainhomotopy group. One might hope that $H_n(X;G) \cong H_n(X) \otimes G$, but this is usually not the case. The next theorem allows one to compute homology with coefficients from unadorned homology, and hence is called a universal coefficient theorem. Afterwards, we shall give the dual result for cohomology.

Theorem 3.1 (Universal Coefficient Theorem for Homology):*Let R be right hereditary, Aan R-module, and (K,d) a complex of projective R-modules. There is a split exact sequence*

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$$0 \to H_n(K) \otimes_R A \xrightarrow{\lambda} H_n(K \otimes_R A) \xrightarrow{\mu} \mathrm{Tor}_1^R(H_{n-1}(K), A) \to 0$$

in which λ and μ are natural. Thus,

$$H_n(K \otimes_R A) \cong H_n(K) \otimes_R A \oplus \mathrm{Tor}_1^R(H_{n-1}(K), A).$$

Remark: The theorem is true if **K** is a complex of flat modules. This, and much more, is proved in the Kunneth formula.

Proof: For each n, there are exact sequences

(*) $$0 \to Z_n(K) \xrightarrow{in} K_n \xrightarrow{dn} B_{n-1}(K) \to 0$$

And

$$0 \to B_{n-1}(K) \to Z_n(K) \to H_{n-1}(K) \to 0$$

(the first is just the definition of cycles and boundaries; the second is just the definition of homology). Splice these two sequences together to obtain an exact sequence

(**) $$0 \to Z_n \to K_n \xrightarrow{d_n} Z_{n-1} \to H_{n-1} \to 0, \qquad K_n \searrow B_{n-1} \nearrow Z_{n-1}$$

Since every K_n is projective and R is hereditary, These shows the submodules $Z_n(\text{of } K_n)$ and $B_{n-1}(\text{of } K_{n-1})$ are also projective. There are two consequences: the exact sequence (*) is split the exact sequence (**) is a projective resolution of H_{n-1}.

$$L\otimes A=0\to Z_n\otimes A\xrightarrow{i_n\otimes 1}K_n\otimes A\xrightarrow{d_n\otimes 1}Z_{n-1}\otimes A\to 0$$

is a complex with homology

$$H_j(L\otimes A)=\mathrm{Tor}_j^R(H_{n-1,}A).$$

Now $\mathrm{Tor}_2^R(H_{n-1},A)=0$ whence $i\otimes 1$is monic (alternatively, that (*) is split implies $Z_n\otimes A$is even a summand of $K_n\otimes A$with injection $i\otimes 1$).We can thus identify $Z_n\otimes A(\text{via } i\otimes 1)$ with a submodule of $K_n\otimes A$.The remaining computations are:

$$\mathrm{Tor}_1^R(H_{n-1},A)=H_1(I\otimes A)=(\ker d_n\otimes 1)/Z_n\otimes A;$$

(***) $$H_{n-1}\otimes A=\mathrm{Tor}_0^R t(H_{n-1},A)=H_0(I\otimes A)=Z_{n-1}\otimes A/\mathrm{im}(d_n\otimes 1).$$

Consider now $K_{n+1}\xrightarrow{d_{n-1}}K_n\xrightarrow{d_n}K_{n-1}$.Examining elements, one verifies the inclusions

$$\mathrm{im}\, d_{n+1}\otimes 1\subset Z_n\otimes A\subset \ker d_n\otimes 1\subset K_n\otimes A.$$

The Third Isomorphism Theorem gives

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$$(\ker d_n \otimes 1 \text{ im} d_{n+1} \otimes 1)/[(Z_n \otimes A)/\text{im } d_{n+1} \otimes 1] \cong \ker d_n \otimes 1/Z_n \otimes A.$$

which maybe rewritten as an exact sequence

$$0 \to Z_n \otimes A/imd_{n+1} \otimes 1 \xrightarrow{\lambda} \ker d_n \otimes 1/imd_{n+1} \otimes 1 \xrightarrow{\mu} \ker d_n \otimes 1/Z_n \otimes A \to 0.$$

The middle term is just $H_n(K \otimes A)$, while we have already computed that the first term is $H_n(K) \otimes A$ (item (***) with $n-1$ replaced by n) and the last term is $\text{Tor}_1^R(H_{n-1}(K).A)$. To see that this sequence splits, observe that Z_n is a summand of K_n (for(*) splits), so that $Z_n \otimes A$ is a summand of $K_n \otimes A$ and hence of $\ker d_n \otimes 1$; it follows that $Z_n \otimes A/\text{im } d_{n+1} \otimes 1$ is a summand of $\ker d_n \otimes 1/\text{im} d_{n+1} \otimes 1.$

Corollary 3.2. If *Xis a topological space and G an abelian group, then for all n,*

$$H_n(X;G) \cong H_n(X) \otimes_Z G \oplus \text{Tor}_1^Z(H_{n-1}(X), G).$$

Proof: By definition, $H_n(X) = H_n(S(X)$ and $H_n(X;G) = H_n(S(X) \otimes G)$. The Universal Coefficient Theorem applies at once, for S(*X*) is a complex of free abelian groups.

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Theorem 3.3 (Universal Coefficient Theorem for Cohomology): *Let R be hereditary, Aan R-module, and (K,d) a complex of projective R-modules. There is a split exact sequence*

$$0 \to \mathrm{Ext}_R^1(H_{n-1}(K), A) \xrightarrow{\lambda} H^n(\mathrm{Hom}_R(K, A)) \xrightarrow{\mu} \mathrm{Hom}_R(H_n(K), A) \to 0$$

in which λ and μ are natural. Thus

$$H^n(\mathrm{Hom}_R(K, A) \cong \mathrm{Hom}_R(H_n(K), A) \oplus \mathrm{Ext}_R^1(H_{4-1}(K), A).$$

Proof: The proof of Theorem 5.11 applies here: the only change is that one now uses the contravariant functorHom$_R$ (, As instead of the covariant functor $\otimes_R A$.

The next result shows that the homology groups $H_n(X)$ of a space X determine its cohomology groups.

Corollary 3.4*If X is topological space and G an abelian group, then for all **n**,*

$$H^n(X; G) \cong \mathrm{Hom}_Z(H_n(X), G) \oplus \mathrm{Ext}_Z^1(H_{n-1}(X), G).$$

It is known that for any sequence of abelian groups $A_0, A_{1,} A_2 ....,$ there exists a topological space X with $H_n(X) \cong A_n$ for all *n*. In contrast, if one defines

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$H^n(X) = H_{-n}(\mathrm{Hom}(S(X), Z)) = H^n(X; Z)$, Nunke-Rotman [1962] prove that if $H^n(X)$ is countable, that it is a sum of a finite group and a free abelian group.

Corollary 3.5 *Let* K *be a complex of free abelian groups. If either* $H_{n-1}(K)$ *is free or A is divisible, then*

$$H^n(Hom_z(K, A)) \cong \mathrm{Hom}_z(H_n(K), A).$$

Proof: Either hypothesis forces $\mathrm{Ext}_z^1(H_{n-1}, A) = 0$.

Of course, variations on this theme are played by assuming other hypotheses guaranteeing that Ext^1 vanish.

Corollary 3.6 *If* K *is a complex of vector space over a field R, and V is a vector space over R, then for all n*

$$H^n(Hom_R(K, V)) \cong \mathrm{Hom}_R(H_n(K), V).$$

In particular,

$$H^n(\mathrm{Hom}_R(\mathrm{K}, R)) \cong H_n(K)^*.$$

Where denotes dual space.*

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