International Journal of Engineering & Scientific Research
Vol. 6 Issue 8, August 2018,
ISSN: 2347-6532 Impact Factor: 6.660
Journal Homepage: http://www.ijmra.us, Email: editorijmie@gmail.com
Double-Blind Peer Reviewed Refereed Open Access International Journal - Included in the International Serial Directories Indexed & Listed at: Ulrich's Periodicals Directory ©, U.S.A., Open J-Gage as well as in Cabell's Directories of Publishing Opportunities, U.S.A

A Purchasing, Inventory Model for Different Deteriorations under Permit Delay in Payments and Price Discount

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Article Received: 10th July, 2018 Article Revised: 22th July, 2018 Article Accepted: 5th August, 2018

Abstract
This paper deals with an inventory model to determine the optimal cycle time and optimal ordering quantity. Here we believe non-instantaneous deteriorating items. The model for deteriorating items is prepared when the deterioration rate is different during the cycle with permissible delay in payments and price discount. Demand is taken as an exponential part of time. Shortages are allowed. Numerical models are given to illustrate the example.

Keywords: Deteriorating items, Delay in payments, Price discount, Shortage.

1. Introduction
In recent years, many researchers have initiated to analyze various problem related to imperfect production process by devoting their time and effort. In most EOQ models, the assumption is that 100% of purchasing units are of dependable character. This unrealistic assumption may not be valid for any purchasing environment. According to [12] Rosenblatt and Lee (1986), the time of the shift from in - control state to out - of - control state follows an exponential distribution with a mean 1/m, assuming m is the smallest value. [8] Makes (1998) has studied several properties of the optimal production and inspection policies in imperfect production process. Some of the above modeler has showed that defective item can be reworked instantaneously at a cost. In the model of [15]Wang (2004), an imperfect EPQ model for production, which are repaired and sold under a free-repair warranty policy discussed by Yeh et al. [7]Liao (2007) has investigated an imperfect production process that requires production corrections and imperfect maintenance. Two states of production process occur, namely state I (Out-of-control state) and state II (in-control state). In Out-of-control state, the product is not perfect and a part is rejected (reworking is impossible) with a probability 'q'. The product is perfect (good quality) with a probability 1-q. [3] Cheng developed a model of imperfect production quantity by establishing the relationship between demand dependent unit production cost and imperfect production process. [13] Salameh.M.K and Jaber.M.Y (2000) developed an inventory model in which items received are of defective quality and after 100% screening, imperfect items are withdrawn from the inventory and sold at a discounted price. [1] Jaggi et al. developed an inventory model for deteriorating items with imperfect quality under permissible delay in payment. [4]Goyal (1985) was the first proponent for developing an economic order quantity model under the condition of permissible delay in payments. [1]Aggarwal and Jaggi (1995) extended [4] Goyals model (1985) to allow the inventory to have deteriorating items. A cash discount can encourage the customer to pay cash on delivery and reduce the default risk.
A permissible delay in payments is considered as a type of price reduction and it can attract new customers and thereby increase sales. In reality, on the operations management side,

supplier is always willing to provide the retailer either a cash discount or a permissible delay in payments. [10] Palanivel.M & Uthayakumar.R (2014) developed an EOQ Model for Non-Instantaneous Deteriorating Items with Power Demand, Time Dependent Holding Cost, Partial Backlogging and permissible Delay in payments.

Inventory model for non-instantaneous deteriorating items have been an object of study for a long time. Generally the project is such that there is no deterioration initially. After certain time deterioration starts and again after certain time the rate of deterioration increase with time. Here we have used such concepts and developed the inventory model for deteriorating items. In the present work, a deterministic inventory model for non-instantaneous deteriorating items with exponential demand is developed. In this model we have considered two cases. In the first case, inventory reaches zero level before deterioration starts and delay in payments is allowed. In the second case, deterioration takes place with a different deterioration rate under permissible delay in payments and price discount. Shortages are allowed in this model. We have shown the suitable numerical example to illustrate the model.

2. Assumptions and Notations

2.1 Assumptions

1. During $[0,\mu_1]$, the items are perfect.
2. During $[\mu_1,\mu_2]$, the rate of deterioration is constant and is equal to θ.
3. During $[\mu_2,t_0]$, the rate of deterioration is time dependent and is equal to θt.
4. The demand of the product is declining as an exponential function of time.
5. Replenishment rate is infinite.
6. Lead time is zero.
7. Shortages are allowed.
8. The model is developed for a single item.
9. The time horizon is infinite.

2.2 Notations

To develop the mathematical model, the following notations are used.

A -- The ordering cost per order.

Q_1 -- The inventory level initially.

D(t)- The demand rate at any time $t \geq 0$, ae^{bt}, $a>0$, $0<b<1$.

T -- The length of replenishment cycle.

P --The purchasing cost per unit item.

h -- The holding cost per unit per unit time.

θ --The deteriorating rate during $\mu_1 < t < \mu_2$.

θt -- The deteriorating rate during $\mu_2 < t < t_0$.

μ_1-- The deteriorating item are initially started at the time.

t_1 --The time at which the inventory reaches zero.

M -- The permissible period of delay in settling the accounts with the supplier.

SR -- The sales revenue.

P_d -- Price of defective items per unit.

TP -- Total relevant profit per unit time.

c_1 -- The shortage cost per period.

s -- The selling cost of the system.

S -- The shortage quantity per period.

3. Mathematical Formulation

Based upon the above notations and assumptions, the following cases are discussed.

Case: 1 $t_1 \le \mu_1$

In this case the purchasing items are good quality with permissible delay in payments.

Case:2 $\mu_1 < t_1$

In this case purchasing items are non-conforming quality and at end of the perioddelay in payments and price discounts are permitted.

Case: 1 $t_1 \le \mu_1$

The inventory system is developed as follows: Q_1 units of item arrive at the inventory system at the beginning of each cycle. During the time interval $[0,\mu_1]$, the inventory level is decreased only due to demand rate, then inventory level reaches zero at or before $t=\mu_1$.Finally shortages occur due to demand during the time interval $[\mu_1, T]$.

The rate of change of the inventory during the positive stock period $(0, t_1)$ and shortage period (t_1, T) is governed by the following differential equations

$$\frac{dI(t)}{dt} = -ae^{bt} \qquad 0 \le t \le t_1 \qquad ------(1)$$

$$\frac{dI(t)}{dt} = -ae^{bt} t_1 \le t \le T \qquad ------(2)$$

The boundary condition are as follows: $I(0)=Q_1$, $I(t_1)=0$, $I(T)=-S$

$$I(t) = \frac{a}{b}\left(1 - e^{bt}\right) + Q_1 \; 0 \le t \le t_1 \qquad ------(3)$$

$$Q_1 = \frac{a}{b}\left(e^{bt_1} - 1\right) \qquad ------(4)$$

$$I(t) = \left[\frac{a}{b}\left(e^{bT} - e^{bt}\right) - S\right] t_1 \le t \le T \qquad ------(5)$$

$$S = \frac{a}{b}\left(e^{bT} - e^{bt_1}\right) \qquad ------(6)$$

The inventory is available during $0 \le t \le t_1$, hence the holding cost is

$$HC = h\int_0^{t_1} I(t)dt$$

$$HC = h\left\{\frac{a}{b^2}\left(bt_1 - e^{bt_1} + 1\right) + Q_1 t_1\right\} \qquad ------(7)$$

The ordering cost per cycle is OC =A --------------- (8)

Shortage due to stock out is accumulated in the system during the inventory (t_1,T), the optimum level of shortage is at t=T, Therefore the total shortage cost during this time period is as follows

$$SC = c_1\int_{t_1}^{T} I(t)dt$$

$$SC = c_1\left[\frac{a}{b}\left((T - t_1)e^{bT} - \frac{1}{b}\left(e^{bT} - e^{bt_1}\right)\right) - S(T - t_1)\right] \qquad -----(9)$$

The purchasing cost is PC= P* $(Q_1 + S)$

$$PC = P\frac{a}{b}\left(e^{bT} - 1\right) \quad ----(10)$$

The sales revenue is

$$SR = s\int_0^T D(t)dt + P_d\theta Q_1$$

$$SR = \frac{sa}{b}\left(e^{bT} - 1\right) + P_d\theta Q_1 \quad ----(11)$$

Case: 1.1M ≤ t_1

In this case, the customer earns interest on the sales revenue up to the permissible delay period and the interest is payable during this period for the items kept in stock

The interest earned per cycle is

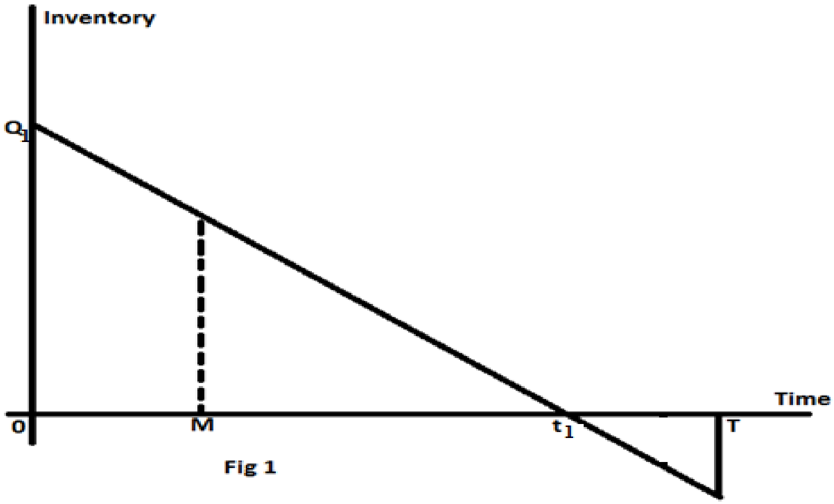

Fig 1

$$IE_1 = sI_e\int_0^M D(t)dt$$

$$IE_1 = sI_e\frac{a}{b}\left(e^{bM} - 1\right) \quad ----(12)$$

The interest payable per cycle is

$$IP_1 = PI_p\int_M^{t_1} I(t)dt$$

$$IP_1 = PI_p\left[\left(\frac{a}{b} + Q_1\right)(t_1 - M) - \frac{a}{b^2}\left(e^{bt_1} - e^{bM}\right)\right] \quad ----(13)$$

Case: 1.2 M > t_1

In this case, the period of delay in payments(M) is more than period with positive inventory (t_1), in this case the retailer earns interest on the sales revenue up to the permissible delay period and no interest is payable during the period for the item kept in stock. Interest earned for the time period [0, T] is

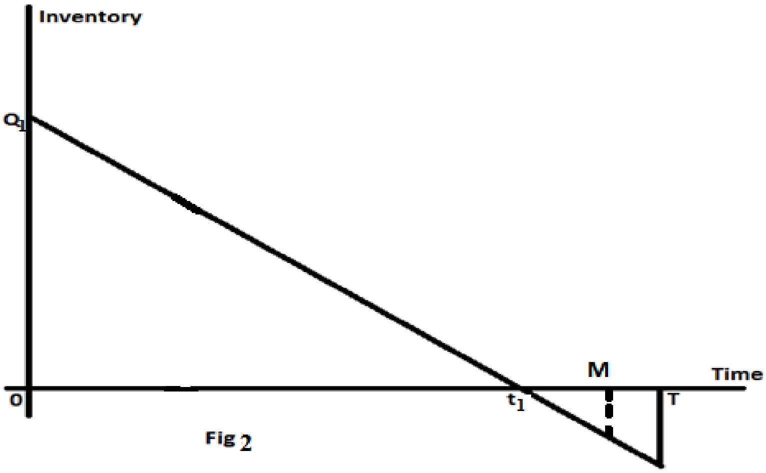

Fig 2

$$IE_2 = sI_e\left[\int_0^{t_1} D(t)dt + a(M - t_1)e^{bt_1}\right]$$

$$IE_2 = sI_e\frac{a}{b}\left[\left(e^{bt_1} - 1\right) + b(M - t_1)e^{bt_1}\right] \qquad ------(14)$$

The interest payable per cycle is $IP_1 = 0 \qquad -------(15)$

Case: 2μ₁<t₁

Here the deteriorating items are available, but different deterioration in the cycle time $(0, t_1)$. The initial inventory level is Q_1 unit at time t=0, from t=0 to $t=\mu_1$ the inventory level reduces only due to demand rate, but during (μ_1, t_1) the inventory is depleted due to both demand and deterioration, until it reaches zero level at time $t=t_1$, during the interval (μ_1, μ_2) inventory depletes due to deterioration at rate θ and delay in payments is allowed. During the interval (μ_2, t_1) inventory depletes due to deterioration at the rate of θt and price discount for deteriorating items is given. The shortages occur during (t_1, T)

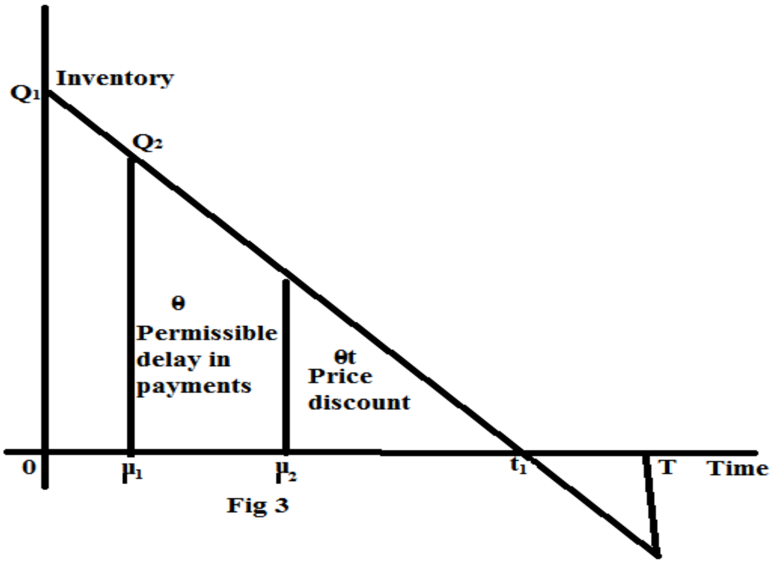

Fig 3

$$\frac{dI(t)}{dt} = -D(t) \qquad 0 \le t \le \mu_1 \qquad -------(16)$$

$$\frac{dI(t)}{dt} + \theta I(t) = -D(t) \qquad \mu_1 \le t \le \mu_2 \qquad -------(17)$$

$$\frac{dI(t)}{dt} + \theta t I(t) = -D(t)\mu_2 \le t \le t_1 \qquad -------(18)$$

$$\frac{dI(t)}{dt} = -D(t) t_1 \le t \le T \qquad ------(19)$$

In addition, using the boundary condition at $I(0)=Q_1$, $I(\mu_1)=Q_2$, $I(t_1)=0$and $I(T)=-S$ we obtain the following equations

$$I(t) = \frac{a}{b}\left(1 - e^{bt}\right) + Q_1 \; 0 \le t \le \mu_1 \qquad ------(20)$$

$$I(t) = \frac{a}{b+\theta}\left[(\mu_1 - t)(\theta + b - b\theta\mu_1)\right] + Q_2(1 + \theta(\mu_1 - t))\mu_1 \le t \le \mu_2 \qquad -----(21)$$

$$I(t) = a\left[\left(1 - \frac{\theta t^2}{2}\right)(t_1 - t) + \frac{b}{2}\left(1 - \frac{\theta t^2}{2}\right)\left({t_1}^2 - t^2\right)\right]\mu_2 \le t \le t_1 \qquad ------(22)$$

$$I(t) = \left[\frac{a}{b}\left(e^{bT} - e^{bt}\right) - S\right] t_1 \le t \le T \qquad ------(23)$$

$$Q_2 = \frac{a}{b}\left(e^{bt_1} - e^{b\mu_1}\right) \qquad -----(24)$$

Based on the assumption and descriptions of the model the total profit, include the following elements

The ordering cost is OC=A --------------- (25)

The holding cost is

$$HC = h\int_0^{t_1} I(t)dt$$

$$HC = h\left[\int_0^{\mu_1} I(t)dt + \int_{\mu_1}^{\mu_2} I(t)dt + \int_{\mu_2}^{t_1} I(t)dt\right]$$

$$HC = h\left\{\frac{a}{b^2}\left(b\mu_1 - e^{b\mu_1} + 1\right) + Q_1\mu_1 - \frac{a}{2(b+\theta)}(\mu_2 - \mu_1)^2(\theta + b - b\theta\mu_1) + Q_2\left(\mu_2 - \mu_1 - \frac{\theta}{2}(\mu_2 - \mu_1)^2\right) + a\left[\frac{1}{2}(t_1 - \mu_2)^2 + \frac{b}{6}\left(2{t_1}^3 - 3\,\mu_2{t_1}^2 + {\mu_2}^3\right) - \frac{\theta}{24}\left({t_1}^4 - 4t_1{\mu_2}^3 + 3{\mu_2}^4\right) - \frac{b\theta}{60}\left(2{t_1}^5 - 5\,{\mu_2}^3{t_1}^2 + 3{\mu_2}^5\right)\right]\right\} \qquad ------(26)$$

The purchasing cost is $PC=P*(Q_1+S)$ ----------------- (27)

The shortage cost is

$$SC = c_1\int_{t_1}^{T} I(t)dt$$

$$SC = c_1\left[\frac{a}{b}\left((T - t_1)e^{bT} - \frac{1}{b}\left(e^{bT} - e^{bt_1}\right)\right) - S(T - t_1)\right] \qquad ------(28)$$

To determine the interest earned, there will be three cases.

Case: 2.1 ($0 \le M \le \mu_1$), Case: 2.2 ($\mu_1 \le M \le \mu_2$), Case: 2.3 ($\mu_2 \le M \le t_1$).

Case: 2.1 ($0 \le M \le \mu_1$)

In this case the retailer can earn interest on revenue generated from the sales up to M. The retailer has to arrange money at some specified rate of interest in order to get his remaining stocks financed for the period M to t_1.

Interest earned per cycle is

$$IE_3 = sI_e \int_0^M D(t)dt$$

$$IE_3 = sI_e \frac{a}{b}\left(e^{bM} - 1\right) ------(29)$$

Interest payable per cycle for the inventory not sold after the due period M is

$$IP_3 = PI_p \int_M^{t_1} I(t)dt$$

$$IP_3 = PI_p \left[\int_M^{\mu_1} I(t)dt + \int_{\mu_1}^{\mu_2} I(t)dt + \int_{\mu_2}^{t_1} I(t)dt\right]$$

$$IP_3 = PI_p \left[\left(\frac{a}{b} + Q_1\right)(\mu_1 - M) - \frac{a}{b^2}\left(e^{b\mu_1} - e^{bM}\right) - \frac{a}{2(b+\theta)}(\mu_2 - \mu_1)^2(\theta + b - b\theta\mu_1) + Q_2\left(\mu_2 - \mu_1 - \frac{\theta}{2}(\mu_2 - \mu_1)^2\right) + a\left[\frac{1}{2}(t_1 - \mu_2)^2 + \frac{b}{6}\left(2{t_1}^3 - 3\,\mu_2{t_1}^2 + {\mu_2}^3\right) - \frac{\theta}{24}\left({t_1}^4 - 4t_1{\mu_2}^3 + 3{\mu_2}^4\right) - \frac{b\theta}{60}\left(2{t_1}^5 - 5\,{\mu_2}^3{t_1}^2 + 3{\mu_2}^5\right)\right]\right] ------(30)$$

Case: 2.2 ($\mu_1 \le M \le \mu_2$)

Since the permissible delay in payments say as greater than the period of $(0, \mu_1)$ with no deterioration (μ_1) but less than the period with positive inventory (t_1)

The interest earned per unit time per cycle is

$$IE_4 = sI_e \int_0^M D(t)dt$$

$$IE_4 = sI_e \left[\int_0^{\mu_1} D(t)dt + a(M - \mu_1)e^{b\mu_1}\right]$$

$$IE_4 = sI_e \left[\frac{a}{b}\left(e^{b\mu_1} - 1\right) + ae^{b\mu_1}(M - \mu_1)\right] ------(31)$$

Interest payable per cycle for the inventory not sold after the due period M is

$$IP_4 = PI_p \int_M^{t_1} I(t)dt$$

$$IP_4 = PI_p \left[\int_M^{\mu_2} I(t)dt + \int_{\mu_2}^{t_1} I(t)dt\right]$$

$$IP_4 = PI_p \left[\left(a\left(1 - \frac{b\theta\mu_1}{b+\theta}\right) + Q_2\theta\right)\left(\mu_1\mu_2 - \mu_1 M - \frac{{\mu_2}^2}{2} + \frac{M^2}{2}\right) + Q_2(\mu_2 - M) + a\left[\frac{1}{2}(t_1 - \mu_2)^2 + \frac{b}{6}\left(2{t_1}^3 - 3\,\mu_2{t_1}^2 + {\mu_2}^3\right) - \frac{\theta}{24}\left({t_1}^4 - 4t_1{\mu_2}^2 + 3{\mu_2}^4\right) - \frac{b\theta}{60}\left(2{t_1}^5 - 5\,{\mu_2}^3{t_1}^2 + 3{\mu_2}^5\right)\right]\right] ------(32)$$

Case: 2.3 ($\mu_2 \le M \le t_1$)

In this case the period of delay in payments (M) is more than the period of (0, μ_1) with constant deterioration (μ_2) but less than the period with positive inventory (t_1)

The interest earned per unit time per cycle is

$$IE_5 = sI_e \int_0^M D(t)dt$$

$$IE_5 = sI_e \left[\int_0^{\mu_1} D(t)dt + \int_{\mu_1}^{\mu_2} D(t)dt + a(M-\mu_2)e^{b\mu_2}\right]$$

$$IE_5 = sI_e \frac{a}{b}\left[\left(1+b(M-\mu_2)\right)e^{b\mu_2} - 1\right] ------(33)$$

Interest payable per cycle for the inventory not sold after the due period M is

$$IP_5 = PI_p \int_M^{t_1} I(t)dt$$

$$IP_5 = PI_p a\left[\frac{1}{2}(t_1-M)^2 + \frac{b}{6}(2t_1{}^3 - 3Mt_1{}^2 + M^3) - \frac{\theta}{24}(t_1{}^4 - 4t_1M^2 + 3M^4) - \frac{b\theta}{60}\left(2t_1{}^5 - 5M^3t_1{}^2 + 3M^5\right)\right] ------(34)$$

Case:2.4

During the time interval (μ_2, t_1) the inventory level decreasing to zero, due to demand and time dependent deterioration. At time t=μ_2 to t= t_1 given price discount for deteriorating items. Finally shortage occur due to demand during the time interval (t_1, T)

The price discount is

$$PD = (s-0.2s)\int_{\mu_2}^{t_1} \theta t I(t)dt$$

$$PD = (s-0.2s)\theta\left(\frac{t_1{}^3}{6} + \frac{bt_1{}^4}{8} - \frac{\theta t_1{}^5}{40} - \frac{b\theta t_1{}^6}{48} - \frac{\mu_2{}^2}{2}\left(t_1 + \frac{bt_1{}^2}{2} - \frac{2\mu_2}{3}\right) + \frac{\mu_2{}^4}{8}\left(b + \theta t_1 + \frac{bt_1{}^2}{2}\right) - \frac{\mu_2{}^5}{2}\left(\frac{1}{5} + \frac{b\mu_2}{12}\right)\right) ------(35)$$

Therefore, the annual total revenue, cost per unit time is

$$TP_1 = \frac{1}{T}[SR - OC - PC - HC - SC - IP_1 + IE_1] ------(36)$$

$$TP_1 = \frac{1}{T}\left\{\frac{sa}{b}\left(e^{bT}-1\right) + P_d\theta Q_1 - A - P(Q_1+S) - h\left(\frac{a}{b^2}\left(bt_1 - e^{bt_1} + 1\right) + t_1Q_1\right) - c_1\left[\frac{a}{b}\left((T-t_1)e^{bT} - \frac{1}{b}\left(e^{bT} - e^{bt_1}\right)\right) - S(T-t_1)\right] - PI_p\left[\left(\frac{a}{b} + Q_1\right)(t_1 - M) - \frac{1}{b^2}\left(e^{bt_1} - e^{bM}\right)\right] + sI_e\frac{a}{b}\left(e^{bM}-1\right)\right\}$$

$$TP_1 = \frac{1}{T}\left\{\frac{sa}{b}\left(e^{bT}-1\right) + P_d\theta\frac{a}{b}\left(e^{bt_1}-1\right) - A - P\frac{a}{b}\left(e^{bT}-1\right) - h\frac{a}{b}\left(\frac{1}{b}\left(1-e^{bt_1}\right) + t_1e^{bt_1}\right) - c_1\frac{a}{b}\left[\left((T-t_1)e^{bt_1} - \frac{1}{b}\left(e^{bT}-e^{bt_1}\right)\right)\right] - PI_p\frac{a}{b}\left[e^{bt_1}(t_1-M) - \frac{1}{b}\left(e^{bt_1}-e^{bM}\right)\right] + sI_e\frac{a}{b}\left(e^{bM}-1\right)\right\}$$

$$TP_2 = \frac{1}{T}[SR - OC - PC - HC - SC - IP_2 + IE_2] \qquad ----\,--(37)$$

$$TP_2 = \frac{1}{T}\Big\{\frac{sa}{b}\left(e^{bT}-1\right) + P_d\theta Q_1 - A - P(Q_1+S) - h\left\{\frac{a}{b^2}\left(bt_1 - e^{bt_1}+1\right) + t_1Q_1\right\} - c_1\left[\frac{a}{b}\left((T-t_1)e^{bT} - \frac{1}{b}\left(e^{bT}-e^{bt_1}\right)\right) - S(T-t_1)\right] + sI_e\frac{a}{b}\left[\left(e^{bt_1}-1\right) + b(M-t_1)e^{bt_1}\right]\Big\}$$

$$TP_2 = \frac{1}{T}\Big\{\frac{sa}{b}\left(e^{bT}-1\right) + P_d\theta\frac{a}{b}\left(e^{bt_1}-1\right) - A - P\frac{a}{b}\left(e^{bT}-1\right) - h\frac{a}{b}\left(\frac{1}{b}\left(1-e^{bt_1}\right) + t_1e^{bt_1}\right) - c_1\frac{a}{b}\left[\left((T-t_1)e^{bt_1} - \frac{1}{b}\left(e^{bT}-e^{bt_1}\right)\right)\right] + sI_e\frac{a}{b}\left[\left(e^{bt_1}-1\right) + b(M-t_1)e^{bt_1}\right]\Big\}$$

$$TP_3 = \frac{1}{T}[SR - OC - PC - HC - SC - IP_3 + IE_3] \qquad ----\,--(38)$$

$$TP_3 = \frac{1}{T}\Big\{\frac{sa}{b}\left(e^{bT}-1\right) + P_d\theta Q_1 - A - P(Q_1+S) - h\left(\frac{a}{b^2}\left(b\mu_1 - e^{b\mu_1}+1\right) + Q_1\mu_1\right) - (h+PI_e)\Big(-\frac{a}{2(b+\theta)}(\mu_2-\mu_1)^2(\theta+b-b\theta\mu_1) + Q_2\left(\mu_2-\mu_1-\frac{\theta}{2}(\mu_2-\mu_1)^2\right) + a\Big[\frac{1}{2}(t_1-\mu_2)^2 + \frac{b}{6}(2t_1^3 - 3\mu_2t_1^2 + \mu_2^3) - \frac{\theta}{24}(t_1^4 - 4t_1\mu_2^2 + 3\mu_2^4) - \frac{b\theta}{60}\left(2t_1^5 - 5\mu_2^3t_1^2 + 3\mu_2^5\right)\Big]\Big) - c_1\left[\frac{a}{b}\left((T-t_1)e^{bT} - \frac{1}{b}\left(e^{bT}-e^{bt_1}\right)\right) - S(T-t_1)\right] - PI_e\left[\left(\frac{a}{b}+Q_1\right)(\mu_1-M) - \frac{a}{b^2}\left(e^{b\mu_1}-e^{bM}\right)\right] + sI_e\frac{a}{b}\left(e^{bM}-1\right)\Big\}$$

$$TP_3 = \frac{1}{T}\Bigg\{\frac{sa}{b}\left(e^{bT}-1\right) + \theta P_d\frac{a}{b}\left(e^{bt_1}-1\right) - A - P\frac{a}{b}\left(e^{bT}-1\right)$$
$$- h\frac{a}{b}\left(\frac{1}{b}\left(b\mu_1 - e^{b\mu_1}+1\right) + \left(e^{bt_1}-1\right)\mu_1\right)$$
$$-(h+PI_e)\Bigg(-\frac{a}{2(b+\theta)}(\mu_2-\mu_1)^2(\theta+b-b\theta\mu_1)$$
$$+\frac{a}{b}\left(e^{bt_1}-e^{b\mu_1}\right)\left(\mu_2-\mu_1-\frac{\theta}{2}(\mu_2-\mu_1)^2\right)$$
$$+a\Bigg[\frac{1}{2}(t_1-\mu_2)^2+\frac{b}{6}(2{t_1}^3-3\,\mu_2{t_1}^2+{\mu_2}^3)-\frac{\theta}{24}({t_1}^4-4t_1{\mu_2}^3+3{\mu_2}^4)$$
$$-\frac{b\theta}{60}\left(2{t_1}^5-5\,{\mu_2}^3{t_1}^2+3{\mu_2}^5\right)\Bigg]\Bigg)$$
$$-c_1\frac{a}{b}\left[\left((T-t_1)e^{bt_1}-\frac{1}{b}\left(e^{bT}-e^{bt_1}\right)\right)\right]$$
$$-PI_e\frac{a}{b}\left[e^{bt_1}(\mu_1-M)-\frac{1}{b}\left(e^{b\mu_1}-e^{bM}\right)\right]+sI_e\frac{a}{b}\left(e^{bM}-1\right)\Bigg\}$$

$$TP_4 = \frac{1}{T}[SR-OC-PC-HC-SC-IP_4+IE_4] \qquad ----\,--(39)$$

$$TP_4 = \frac{1}{T}\Bigg\{\frac{sa}{b}\left(e^{bT}-1\right)+P_d dQ_1 - A - P(Q_1+S)$$
$$-h\Bigg(\frac{a}{b^2}\left(b\mu_1-e^{b\mu_1}+1\right)+Q_1\mu_1-\frac{a}{2(b+\theta)}(\mu_2-\mu_1)^2(\theta+b-b\theta\mu_1)$$
$$+Q_2\left(\mu_2-\mu_1-\frac{\theta}{2}(\mu_2-\mu_1)^2\right)\Bigg)$$
$$-(h+PI_e)a\Bigg[\frac{1}{2}(t_1-\mu_2)^2+\frac{b}{6}(2{t_1}^3-3\,\mu_2{t_1}^2+{\mu_2}^3)$$
$$-\frac{\theta}{24}({t_1}^4-4t_1{\mu_2}^2+3{\mu_2}^4)-\frac{b\theta}{60}\left(2{t_1}^5-5\,{\mu_2}^3{t_1}^2+3{\mu_2}^5\right)\Bigg]$$
$$-c_1\left[\frac{a}{b}\left((T-t_1)e^{bT}-\frac{1}{b}\left(e^{bT}-e^{bt_1}\right)\right)-S(T-t_1)\right]$$
$$-PI_e\Bigg[\left(a\left(1-\frac{b\theta\mu_1}{b+\theta}\right)+Q_2\theta\right)\left(\mu_1\mu_2-\mu_1 M-\frac{{\mu_2}^2}{2}+\frac{M^2}{2}\right)$$
$$+Q_2(\mu_2-M)\Bigg]sI_e\left(\frac{a}{b}\left(e^{b\mu_1}-1\right)+ae^{b\mu_1}(M-\mu_1)\right)\Bigg\}$$

$$TP_4 = \frac{1}{T}\left\{\frac{sa}{b}\left(e^{bT}-1\right)+\theta P_d\frac{a}{b}\left(e^{bt_1}-1\right)-A-P\frac{a}{b}\left(e^{bT}-1\right)\right.$$
$$-h\frac{a}{b}\left(\frac{1}{b}\left(b\mu_1-e^{b\mu_1}+1\right)+\left(e^{bt_1}-1\right)\mu_1\right.$$
$$-\frac{a}{2(b+\theta)}(\mu_2-\mu_1)^2(\theta+b-b\theta\mu_1)$$
$$\left.+\frac{a}{b}\left(e^{bt_1}-e^{b\mu_1}\right)\left(\mu_2-\mu_1-\frac{\theta}{2}(\mu_2-\mu_1)^2\right)\right)$$
$$-(h+PI_e)a\left(\frac{1}{2}(t_1-\mu_2)^2+\frac{b}{6}\left(2{t_1}^3-3\,\mu_2{t_1}^2+{\mu_2}^3\right)\right.$$
$$\left.-\frac{\theta}{24}\left({t_1}^4-4t_1{\mu_2}^3+3{\mu_2}^4\right)-\frac{b\theta}{60}\left(2{t_1}^5-5\,{\mu_2}^3{t_1}^2+3{\mu_2}^5\right)\right)$$
$$-c_1\frac{a}{b}\left[\left((T-t_1)e^{bt_1}-\frac{1}{b}\left(e^{bT}-e^{bt_1}\right)\right)\right]$$
$$-PI_e\left[\left(a\left(1-\frac{b\theta\mu_1}{b+\theta}\right)+\frac{a}{b}\left(e^{bt_1}-e^{b\mu_1}\right)\theta\right)\left(\mu_1\mu_2-\mu_1 M-\frac{{\mu_2}^2}{2}+\frac{M^2}{2}\right)\right.$$
$$\left.\left.+\frac{a}{b}\left(e^{bt_1}-e^{b\mu_1}\right)(\mu_2-M)\right]+sI_e\left(\frac{a}{b}\left(e^{b\mu_1}-1\right)+ae^{b\mu_1}(M-\mu_1)\right)\right\}$$

$$TP_5 = \frac{1}{T}\left[SR-OC-PC-HC-SC-IP_4+IE_4\right] \qquad ----(40)$$

$$TP_5 = \frac{1}{T}\left\{\frac{sa}{b}\left(e^{bT}-1\right)+P_d dQ_1-A-P(Q_1+S)\right.$$
$$-h\left(\frac{a}{b^2}\left(b\mu_1-e^{b\mu_1}+1\right)+Q_1\mu_1-\frac{a}{2(b+\theta)}(\mu_2-\mu_1)^2(\theta+b-b\theta\mu_1)\right.$$
$$+Q_2\left(\mu_2-\mu_1-\frac{\theta}{2}(\mu_2-\mu_1)^2\right)$$
$$+a\left[\frac{1}{2}(t_1-\mu_2)^2+\frac{b}{6}\left(2{t_1}^3-3\,\mu_2{t_1}^2+{\mu_2}^3\right)-\frac{\theta}{24}\left({t_1}^4-4t_1{\mu_2}^2+3{\mu_2}^4\right)\right.$$
$$\left.\left.-\frac{b\theta}{60}\left(2{t_1}^5-5\,{\mu_2}^3{t_1}^2+3{\mu_2}^5\right)\right]\right)$$
$$-c_1\left[\frac{a}{b}\left((T-t_1)e^{bT}-\frac{1}{b}\left(e^{bT}-e^{bt_1}\right)\right)-S(T-t_1)\right]$$
$$-PI_p a\left[\frac{1}{2}(t_1-M)^2+\frac{b}{6}\left(2{t_1}^3-3M{t_1}^2+M^3\right)-\frac{\theta}{24}\left({t_1}^4-4t_1M^2+3M^4\right)\right.$$
$$\left.\left.-\frac{b\theta}{60}\left(2{t_1}^5-5M^3{t_{11}}^2+3M^5\right)\right]+sI_e\frac{a}{b}\left[1+b(M-\mu_2)e^{b\mu_2}-1\right]\right\}$$

$$TP_5 = \frac{1}{T}\left\{\frac{sa}{b}\left(e^{bT}-1\right) + \theta P_d\frac{a}{b}\left(e^{bt_1}-1\right) - A - P\frac{a}{b}\left(e^{bT}-1\right)\right.$$

$$- h\left(\frac{a}{b^2}\left(b\mu_1 - e^{b\mu_1}+1\right) + \frac{a}{b}\left(e^{bt_1}-1\right)\mu_1\right.$$

$$- \frac{a}{2(b+\theta)}(\mu_2-\mu_1)^2(\theta + b - b\theta\mu_1)$$

$$+ \frac{a}{b}\left(e^{bt_1}-e^{b\mu_1}\right)\left(\mu_2-\mu_1-\frac{\theta}{2}(\mu_2-\mu_1)^2\right)$$

$$+ a\left(\frac{1}{2}(t_1-\mu_2)^2 + \frac{b}{6}(2{t_1}^3 - 3\,\mu_2{t_1}^2 + {\mu_2}^3) - \frac{\theta}{24}({t_1}^4 - 4t_1{\mu_2}^3 + 3{\mu_2}^4)\right.$$

$$\left.\left. - \frac{b\theta}{60}\left(2{t_1}^5 - 5\,{\mu_2}^3{t_1}^2 + 3{\mu_2}^5\right)\right)\right)$$

$$- c_1\frac{a}{b}\left[\left((T-t_1)e^{bt_1} - \frac{1}{b}\left(e^{bT}-e^{bt_1}\right)\right)\right]$$

$$- PI_p a\left[\frac{1}{2}(t_1-M)^2 + \frac{b}{6}(2{t_1}^3 - 3M{t_1}^2 + M^3) - \frac{\theta}{24}({t_1}^4 - 4t_1M^2 + 3M^4)\right.$$

$$\left.\left. - \frac{b\theta}{60}\left(2{t_1}^5 - 5M^3{t_{11}}^2 + 3M^5\right)\right] + sI_e\frac{a}{b}\left[1 + b(M-\mu_2)e^{b\mu_2} - 1\right]\right\}$$

$$TP_6 = \frac{1}{T}[SR - OC - PC - HC - SC - PD] \qquad ----\,--(41)$$

$$TP_6 = \frac{1}{T}\left\{\frac{sa}{b}\left(e^{bT}-1\right) + P_d dQ_1 - A - P(Q_1+S)\right.$$

$$- h\left(\frac{a}{b^2}\left(b\mu_1 - e^{b\mu_1}+1\right) + Q_1\mu_1 - \frac{a}{2(b+\theta)}(\mu_2-\mu_1)^2(\theta + b - b\theta\mu_1)\right.$$

$$+ Q_2\left(\mu_2-\mu_1-\frac{\theta}{2}(\mu_2-\mu_1)^2\right)$$

$$+ a\left[\frac{1}{2}(t_1-\mu_2)^2 + \frac{b}{6}(2{t_1}^3 - 3\,\mu_2{t_1}^2 + {\mu_2}^3) - \frac{\theta}{24}({t_1}^4 - 4t_1{\mu_2}^2 + 3{\mu_2}^4)\right.$$

$$\left.\left. - \frac{b\theta}{60}\left(2{t_1}^5 - 5\,{\mu_2}^3{t_1}^2 + 3{\mu_2}^5\right)\right]\right)$$

$$- c_1\left[\frac{a}{b}\left((T-t_1)e^{bT} - \frac{1}{b}\left(e^{bT}-e^{bt_1}\right)\right) - S(T-t_1)\right]$$

$$- (s-0.2s)\theta\left(\frac{{t_1}^3}{6} + \frac{b{t_1}^4}{8} - \frac{\theta{t_1}^5}{40} - \frac{b\theta{t_1}^6}{48} - \frac{{\mu_2}^2}{2}\left(t_1 + \frac{b{t_1}^2}{2} - \frac{2\mu_2}{3}\right)\right.$$

$$\left.\left. + \frac{{\mu_2}^4}{8}\left(b + \theta t_1 + \frac{b{t_1}^2}{2}\right) - \frac{{\mu_2}^5}{2}\left(\frac{1}{5} + \frac{b\mu_2}{12}\right)\right)\right\}$$

$$TP_6=\frac{1}{T}\left\{\frac{sa}{b}\left(e^{bT}-1\right)+\theta P_d\frac{a}{b}\left(e^{bt_1}-1\right)-A-P\frac{a}{b}\left(e^{bT}-1\right)\right.$$
$$-h\left(\frac{a}{b^2}\left(b\mu_1-e^{b\mu_1}+1\right)+\frac{a}{b}\left(e^{bt_1}-1\right)\mu_1\right.$$
$$-\frac{a}{2(b+\theta)}(\mu_2-\mu_1)^2(\theta+b-b\theta\mu_1)$$
$$+\frac{a}{b}\left(e^{bt_1}-e^{b\mu_1}\right)\left(\mu_2-\mu_1-\frac{\theta}{2}(\mu_2-\mu_1)^2\right)$$
$$+a\left(\frac{1}{2}(t_1-\mu_2)^2+\frac{b}{6}(2{t_1}^3-3\,\mu_2{t_1}^2+{\mu_2}^3)-\frac{\theta}{24}({t_1}^4-4t_1{\mu_2}^3+3{\mu_2}^4)\right.$$
$$\left.\left.-\frac{b\theta}{60}\left(2{t_1}^5-5\,{\mu_2}^3{t_1}^2+3{\mu_2}^5\right)\right)\right)$$
$$-c_1\frac{a}{b}\left[\left((T-t_1)e^{bt_1}-\frac{1}{b}\left(e^{bT}-e^{bt_1}\right)\right)\right]$$
$$-(s-0.2s)\theta\left(\frac{{t_1}^3}{6}+\frac{b{t_1}^4}{8}-\frac{\theta{t_1}^5}{40}-\frac{b\theta{t_1}^6}{48}-\frac{{\mu_2}^2}{2}\left(t_1+\frac{b{t_1}^2}{2}-\frac{2\mu_2}{3}\right)\right.$$
$$\left.\left.+\frac{{\mu_2}^4}{8}\left(b+\theta t_1+\frac{b{t_1}^2}{2}\right)-\frac{{\mu_2}^5}{2}\left(\frac{1}{5}+\frac{b\mu_2}{12}\right)\right)\right\}$$

To maximize the total profit (TP) per unit time, the optimal value of T and t_1 can be obtained by solving the following equations

$$\frac{\partial TP_i}{\partial T}=0,\frac{\partial TP_i}{\partial t_1}=0 \qquad i=1,2,3,4,5,6. \qquad ----(42)$$

$$\frac{\partial TP_1}{\partial T}=-\frac{1}{T^2}\left\{\frac{sa}{b}\left(e^{bT}(1-bT)-1\right)+P_d\theta\frac{a}{b}\left(e^{bt_1}-1\right)\ -A-P\frac{a}{b}\left(e^{bT}(1-bT)-1\right)\right.$$
$$-h\frac{a}{b}\left(\frac{1}{b}\left(1-e^{bt_1}\right)+t_1e^{bt_1}\right)-c_1\frac{a}{b}\left(\left(\frac{1}{b}-t_1\right)e^{bt_1}+\frac{1}{b}(bT-1)e^{bT}\right)$$
$$\left.-PI_p\frac{a}{b}\left[e^{bt_1}(t_1-M)-\frac{1}{b}\left(e^{bt_1}-e^{bM}\right)\right]+sI_e\frac{a}{b}\left(e^{bM}-1\right)\right\}$$

$$\frac{\partial^2 TP_1}{\partial T^2}=\frac{2}{T^3}\left\{\frac{sa}{b}\left(e^{bT}(1-bT)-1\right)+P_d\theta\frac{a}{b}\left(e^{bt_1}-1\right)\ -A-P\frac{a}{b}\left(e^{bT}(1-bT)-1\right)\right.$$
$$-h\frac{a}{b}\left(\frac{1}{b}\left(1-e^{bt_1}\right)+t_1e^{bt_1}\right)-c_1\frac{a}{b}\left(\left(\frac{1}{b}-t_1\right)e^{bt_1}+\left(T-\frac{1}{b}\right)e^{bT}\right)$$
$$\left.-PI_p\frac{a}{b}\left[e^{bt_1}(t_1-M)-\frac{1}{b}\left(e^{bt_1}-e^{bM}\right)\right]+sI_e\frac{a}{b}\left(e^{bM}-1\right)\right\}$$
$$+\frac{abe^{bT}}{T}(s-P)+\frac{c_1ae^{bT}}{T}$$

$$\frac{\partial TP_1}{\partial t_1}=\frac{1}{T}\left\{P_d\theta ae^{bt_1}-hat_1e^{bt_1}-c_1ae^{bt_1}(T-t_1)-PI_pae^{bt_1}(t_1-M)\right\}$$

$$\frac{\partial^2 TP_1}{\partial {t_1}^2} = \frac{1}{T}\left\{P_d\theta abe^{bt_1} - habt_1e^{bt_1} - c_1ae^{bt_1}(1+bt_1) - PI_pabe^{bt_1}(t_1 - M)\right\}$$

$$\frac{\partial TP_2}{\partial T} = -\frac{1}{T^2}\left\{\frac{sa}{b}\left(e^{bT}(1-bT)-1\right) + P_d\theta\frac{a}{b}\left(e^{bt_1}-1\right) - A - P\frac{a}{b}\left(e^{bT}(1-bT)-1\right) - h\frac{a}{b}\left(\frac{1}{b}\left(1-e^{bt_1}\right)+t_1e^{bt_1}\right) - c_1\frac{a}{b}\left(\left(\frac{1}{b}-t_1\right)e^{bt_1} + \frac{1}{b}(bT-1)e^{bT}\right) + sI_e\frac{a}{b}\left[\left(e^{bt_1}-1\right)+b(M-t_1)e^{bt_1}\right]\right\}$$

$$\frac{\partial^2 TP_2}{\partial T^2} = \frac{2}{T^3}\left\{\frac{sa}{b}\left(e^{bT}(1-bT)-1\right) + P_d\theta\frac{a}{b}\left(e^{bt_1}-1\right) - A - P\frac{a}{b}\left(e^{bT}(1-bT)-1\right) - h\frac{a}{b}\left(\frac{1}{b}\left(1-e^{bt_1}\right)+t_1e^{bt_1}\right) - c_1\frac{a}{b}\left(\left(\frac{1}{b}-t_1\right)e^{bt_1} + \left(T-\frac{1}{b}\right)e^{bT}\right) + sI_e\frac{a}{b}\left[\left(e^{bt_1}-1\right)+b(M-t_1)e^{bt_1}\right]\right\} + \frac{abe^{bT}}{T}(s-P) + \frac{c_1ae^{bT}}{T}$$

$$\frac{\partial TP_2}{\partial t_1} = \frac{1}{T}\left\{P_d\theta ae^{bt_1} - hat_1e^{bt_1} - c_1ae^{bt_1}(T-t_1) + sI_eae^{bt_1}(M-t_1)\right\}$$

$$\frac{\partial^2 TP_2}{\partial {t_1}^2} = \frac{1}{T}\left\{P_d\theta abe^{bt_1} - habt_1e^{bt_1} - c_1ae^{bt_1}(1+bt_1) + sI_eabe^{bt_1}(Mb - bt_1 - 1)\right\}$$

$$\frac{\partial TP_3}{\partial T} = -\frac{1}{T^2}\left\{\frac{sa}{b}\left(e^{bT}(1-bT)-1\right) + P_d\theta\frac{a}{b}\left(e^{bt_1}-1\right) - A - P\frac{a}{b}\left(e^{bT}(1-bT)-1\right) - h\frac{a}{b}\left(\frac{1}{b}\left(b\mu_1 - e^{b\mu_1}+1\right) + \left(e^{bt_1}-1\right)\mu_1\right) - (h+PI_e)\left(-\frac{a}{2(b+\theta)}(\mu_2-\mu_1)^2(\theta+b-b\theta\mu_1) + \frac{a}{b}\left(e^{bt_1}-e^{b\mu_1}\right)\left(\mu_2-\mu_1-\frac{\theta}{2}(\mu_2-\mu_1)^2\right) + a\left[\frac{1}{2}(t_1-\mu_2)^2 + \frac{b}{6}\left(2{t_1}^3 - 3\mu_2{t_1}^2 + {\mu_2}^3\right) - \frac{\theta}{24}\left({t_1}^4 - 4t_1{\mu_2}^3 + 3{\mu_2}^4\right) - \frac{b\theta}{60}\left(2{t_1}^5 - 5{\mu_2}^3{t_1}^2 + 3{\mu_2}^5\right)\right]\right) - c_1\frac{a}{b}\left(\left(\frac{1}{b}-t_1\right)e^{bt_1} + \frac{1}{b}(bT-1)e^{bT}\right) - PI_e\frac{a}{b}\left[e^{bt_1}(\mu_1 - M) - \frac{1}{b}\left(e^{b\mu_1}-e^{bM}\right)\right] + sI_e\frac{a}{b}\left(e^{bM}-1\right)\right\}$$

$$\frac{\partial^2 TP_3}{\partial T^2} = \frac{2}{T^3}\Bigg\{\frac{sa}{b}\left(e^{bT}(1-bT)-1\right) + P_d\theta\frac{a}{b}\left(e^{bt_1}-1\right) - A - P\frac{a}{b}\left(e^{bT}(1-bT)-1\right)$$
$$- h\frac{a}{b}\left(\frac{1}{b}\left(b\mu_1 - e^{b\mu_1}+1\right) + \left(e^{bt_1}-1\right)\mu_1\right)$$
$$- (h+PI_e)\Bigg(-\frac{a}{2(b+\theta)}(\mu_2-\mu_1)^2(\theta+b-b\theta\mu_1)$$
$$+\frac{a}{b}\left(e^{bt_1}-e^{b\mu_1}\right)\left(\mu_2-\mu_1-\frac{\theta}{2}(\mu_2-\mu_1)^2\right)$$
$$+ a\Bigg[\frac{1}{2}(t_1-\mu_2)^2 + \frac{b}{6}\left(2{t_1}^3 - 3\,\mu_2{t_1}^2 + {\mu_2}^3\right) - \frac{\theta}{24}\left({t_1}^4 - 4t_1{\mu_2}^3 + 3{\mu_2}^4\right)$$
$$-\frac{b\theta}{60}\left(2{t_1}^5 - 5\,{\mu_2}^3{t_1}^2 + 3{\mu_2}^5\right)\Bigg]\Bigg) - c_1\frac{a}{b}\left(\left(\frac{1}{b}-t_1\right)e^{bt_1} + \left(T-\frac{1}{b}\right)e^{bT}\right)$$
$$- PI_e\frac{a}{b}\left[e^{bt_1}(\mu_1-M) - \frac{1}{b}\left(e^{b\mu_1}-e^{bM}\right)\right] + sI_e\frac{a}{b}\left(e^{bM}-1\right)\Bigg\}$$
$$+\frac{abe^{bT}}{T}(s-P) + \frac{c_1ae^{bT}}{T}$$

$$\frac{\partial TP_3}{\partial t_1} = \frac{1}{T}\Bigg\{P_d\theta\alpha e^{bt_1} - hat_1e^{bt_1}$$
$$-(h+PI_e)\Bigg(ae^{bt_1}\left(\mu_2-\mu_1-\frac{\theta}{2}(\mu_2-\mu_1)^2\right)$$
$$+ a\left[(t_1-\mu_2) + b({t_1}^2 - \mu_2t_1) - \frac{\theta}{6}\left({t_1}^3-{\mu_2}^3\right) - \frac{b\theta}{6}\left({t_1}^4 - {\mu_2}^3t_1\right)\right]\Bigg)$$
$$- c_1ae^{bt_1}(T-t_1) - PI_e\alpha e^{bt_1}(\mu_1-M)\Bigg\}$$

$$\frac{\partial^2 TP_3}{\partial {t_1}^2} = \frac{1}{T}\Bigg\{P_d\theta abe^{bt_1} - habt_1e^{bt_1}$$
$$-(h+PI_e)\Bigg(abe^{bt_1}\left(\mu_2-\mu_1-\frac{\theta}{2}(\mu_2-\mu_1)^2\right)$$
$$+\left[a + b(2t_1-\mu_2) - \frac{\theta}{6}\left(3{t_1}^2\right) - \frac{b\theta}{6}\left(4{t_1}^3-{\mu_2}^3\right)\right]\Bigg) - c_1ae^{bt_1}(1-bt_1)$$
$$- PI_eabe^{bt_1}(\mu_1-M)\Bigg\}$$

$$\frac{\partial TP_4}{\partial T} = -\frac{1}{T^2}\Bigg\{\frac{sa}{b}\left(e^{bT}(1-bT)-1\right) + P_d\theta\frac{a}{b}\left(e^{bt_1}-1\right) - A - P\frac{a}{b}\left(e^{bT}(1-bT)-1\right)$$
$$-h\Bigg(\frac{a}{b^2}\left(b\mu_1 - e^{b\mu_1}+1\right) + \frac{a}{b}\left(e^{bt_1}-1\right)\mu_1$$
$$-\frac{a}{2(b+\theta)}(\mu_2-\mu_1)^2(\theta+b-b\theta\mu_1)$$
$$+\frac{a}{b}\left(e^{bt_1}-e^{b\mu_1}\right)\left(\mu_2-\mu_1-\frac{\theta}{2}(\mu_2-\mu_1)^2\right)\Bigg)$$
$$-(h+PI_e)a\Bigg(\frac{1}{2}(t_1-\mu_2)^2 + \frac{b}{6}(2{t_1}^3 - 3\,\mu_2{t_1}^2 + {\mu_2}^3)$$
$$-\frac{\theta}{24}({t_1}^4 - 4t_1{\mu_2}^3 + 3{\mu_2}^4) - \frac{b\theta}{60}\left(2{t_1}^5 - 5\,{\mu_2}^3{t_1}^2 + 3{\mu_2}^5\right)\Bigg)$$
$$-c_1\frac{a}{b}\left(\left(\frac{1}{b}-t_1\right)e^{bt_1} + \frac{1}{b}(bT-1)e^{bT}\right)$$
$$-PI_e\Bigg[\left(a\left(1-\frac{b\theta\mu_1}{b+\theta}\right) + \frac{a}{b}\left(e^{bt_1}-e^{b\mu_1}\right)\theta\right)\left(\mu_1\mu_2 - \mu_1 M - \frac{{\mu_2}^2}{2} + \frac{M^2}{2}\right)$$
$$+\frac{a}{b}\left(e^{bt_1}-e^{b\mu_1}\right)(\mu_2 - M)\Bigg] + sI_e\left(\frac{a}{b}\left(e^{b\mu_1}-1\right) + ae^{b\mu_1}(M-\mu_1)\right)\Bigg\}$$

$$\frac{\partial^2 TP_4}{\partial T^2} = \frac{2}{T^3}\Bigg\{\frac{sa}{b}\left(e^{bT}(1-bT)-1\right) + P_d\theta\frac{a}{b}\left(e^{bt_1}-1\right) - A - P\frac{a}{b}\left(e^{bT}(1-bT)-1\right)$$
$$-h\Bigg(\frac{a}{b^2}\left(b\mu_1 - e^{b\mu_1}+1\right) + \frac{a}{b}\left(e^{bt_1}-1\right)\mu_1$$
$$-\frac{a}{2(b+\theta)}(\mu_2-\mu_1)^2(\theta+b-b\theta\mu_1)$$
$$+\frac{a}{b}\left(e^{bt_1}-e^{b\mu_1}\right)\left(\mu_2-\mu_1-\frac{\theta}{2}(\mu_2-\mu_1)^2\right)\Bigg)$$
$$-(h+PI_e)a\Bigg(\frac{1}{2}(t_1-\mu_2)^2 + \frac{b}{6}(2{t_1}^3 - 3\,\mu_2{t_1}^2 + {\mu_2}^3)$$
$$-\frac{\theta}{24}({t_1}^4 - 4t_1{\mu_2}^3 + 3{\mu_2}^4) - \frac{b\theta}{60}\left(2{t_1}^5 - 5\,{\mu_2}^3{t_1}^2 + 3{\mu_2}^5\right)\Bigg)$$
$$-c_1\frac{a}{b}\left(\left(\frac{1}{b}-t_1\right)e^{bt_1} + \left(T-\frac{1}{b}\right)e^{bT}\right)$$
$$-PI_e\Bigg[\left(a\left(1-\frac{b\theta\mu_1}{b+\theta}\right) + \frac{a}{b}\left(e^{bt_1}-e^{b\mu_1}\right)\theta\right)\left(\mu_1\mu_2 - \mu_1 M - \frac{{\mu_2}^2}{2} + \frac{M^2}{2}\right)$$
$$+\frac{a}{b}\left(e^{bt_1}-e^{b\mu_1}\right)(\mu_2 - M)\Bigg] + sI_e\left(\frac{a}{b}\left(e^{b\mu_1}-1\right) + ae^{b\mu_1}(M-\mu_1)\right)\Bigg\}$$
$$+\frac{abe^{bT}}{T}(s-P) + \frac{c_1 a e^{bT}}{T}$$

$$\frac{\partial TP_4}{\partial t_1} = \frac{1}{T}\left\{P_d\theta a e^{bt_1} - h\left(a\mu_1 e^{bt_1} + ae^{bt_1}\left(\mu_2 - \mu_1 - \frac{\theta}{2}(\mu_2 - \mu_1)^2\right)\right)\right.$$
$$- (h + PI_e)a\left((t_1 - \mu_2) + b({t_1}^2 - \mu_2 t_1) - \frac{\theta}{6}({t_1}^3 - {\mu_2}^3)\right.$$
$$\left. - \frac{b\theta}{6}({t_1}^4 - {\mu_2}^3 t_1)\right)$$
$$- PI_e\left[(ae^{bt_1}\theta)\left(\mu_1\mu_2 - \mu_1 M - \frac{{\mu_2}^2}{2} + \frac{M^2}{2}\right) + ae^{bt_1}(\mu_2 - M)\right]$$
$$\left. - c_1 ae^{bt_1}(T - t_1)\right\}$$

$$\frac{\partial^2 TP_4}{\partial {t_1}^2} = \frac{1}{T}\left\{P_d\theta ab e^{bt_1} - h\left(ab\mu_1 e^{bt_1} + abe^{bt_1}\left(\mu_2 - \mu_1 - \frac{\theta}{2}(\mu_2 - \mu_1)^2\right)\right)\right.$$
$$- (h + PI_e)\left[a + b(2t_1 - \mu_2) - \frac{\theta}{6}(3{t_1}^2) - \frac{b\theta}{6}(4{t_1}^3 - {\mu_2}^3)\right]$$
$$- PI_e\left[(abe^{bt_1}\theta)\left(\mu_1\mu_2 - \mu_1 M - \frac{{\mu_2}^2}{2} + \frac{M^2}{2}\right) + abe^{bt_1}(\mu_2 - M)\right]$$
$$\left. + c_1 ae^{bt_1}(1 + bt_1)\right\}$$

$$\frac{\partial TP_5}{\partial T} = -\frac{1}{T^2}\left\{\frac{sa}{b}\left(e^{bT}(1 - bT) - 1\right) + P_d\theta\frac{a}{b}\left(e^{bt_1} - 1\right) - A - P\frac{a}{b}\left(e^{bT}(1 - bT) - 1\right)\right.$$
$$- h\left(\frac{a}{b^2}\left(b\mu_1 - e^{b\mu_1} + 1\right) + \frac{a}{b}\left(e^{bt_1} - 1\right)\mu_1\right.$$
$$- \frac{a}{2(b + \theta)}(\mu_2 - \mu_1)^2(\theta + b - b\theta\mu_1)$$
$$+ \frac{a}{b}\left(e^{bt_1} - e^{b\mu_1}\right)\left(\mu_2 - \mu_1 - \frac{\theta}{2}(\mu_2 - \mu_1)^2\right)$$
$$+ a\left(\frac{1}{2}(t_1 - \mu_2)^2 + \frac{b}{6}(2{t_1}^3 - 3\mu_2{t_1}^2 + {\mu_2}^3) - \frac{\theta}{24}({t_1}^4 - 4t_1{\mu_2}^3 + 3{\mu_2}^4)\right.$$
$$\left.\left. - \frac{b\theta}{60}\left(2{t_1}^5 - 5{\mu_2}^3{t_1}^2 + 3{\mu_2}^5\right)\right)\right)$$
$$- c_1\frac{a}{b}\left(\left(\frac{1}{b} - t_1\right)e^{bt_1} + \frac{1}{b}(bT - 1)e^{bT}\right)$$
$$- PI_p a\left[\frac{1}{2}(t_1 - M)^2 + \frac{b}{6}(2{t_1}^3 - 3M{t_1}^2 + M^3) - \frac{\theta}{24}({t_1}^4 - 4t_1 M^2 + 3M^4)\right.$$
$$\left.\left. - \frac{b\theta}{60}\left(2{t_1}^5 - 5M^3{t_{11}}^2 + 3M^5\right)\right] + sI_e\frac{a}{b}\left[1 + b(M - \mu_2)e^{b\mu_2} - 1\right]\right\}$$

$$\frac{\partial^2 TP_5}{\partial T^2} = \frac{2}{T^3}\Bigg\{\frac{sa}{b}\left(e^{bT}(1-bT)-1\right)+P_d\theta\frac{a}{b}\left(e^{bt_1}-1\right) - A - P\frac{a}{b}\left(e^{bT}(1-bT)-1\right)$$
$$-h\Bigg(\frac{a}{b^2}\left(b\mu_1-e^{b\mu_1}+1\right)+\frac{a}{b}\left(e^{bt_1}-1\right)\mu_1$$
$$-\frac{a}{2(b+\theta)}(\mu_2-\mu_1)^2(\theta+b-b\theta\mu_1)$$
$$+\frac{a}{b}\left(e^{bt_1}-e^{b\mu_1}\right)\left(\mu_2-\mu_1-\frac{\theta}{2}(\mu_2-\mu_1)^2\right)$$
$$+a\Bigg(\frac{1}{2}(t_1-\mu_2)^2+\frac{b}{6}(2{t_1}^3-3\,\mu_2{t_1}^2+{\mu_2}^3)-\frac{\theta}{24}({t_1}^4-4t_1{\mu_2}^3+3{\mu_2}^4)$$
$$-\frac{b\theta}{60}\left(2{t_1}^5-5\,{\mu_2}^3{t_1}^2+3{\mu_2}^5\right)\Bigg)\Bigg)-c_1\frac{a}{b}\left(\left(\frac{1}{b}-t_1\right)e^{bt_1}+\left(T-\frac{1}{b}\right)e^{bT}\right)$$
$$-PI_p a\left[\frac{1}{2}(t_1-M)^2+\frac{b}{6}(2{t_1}^3-3M{t_1}^2+M^3)-\frac{\theta}{24}({t_1}^4-4t_1M^2+3M^4)\right.$$
$$\left.-\frac{b\theta}{60}\left(2{t_1}^5-5M^3{t_{11}}^2+3M^5\right)\right]+sI_e\frac{a}{b}\left[1+b(M-\mu_2)e^{b\mu_2}-1\right]\Bigg\}$$
$$+\frac{abe^{bT}}{T}(s-P)+\frac{c_1ae^{bT}}{T}$$

$$\frac{\partial TP_5}{\partial t_1} = \frac{1}{T}\Bigg\{P_d\theta ae^{bt_1}-c_1ae^{bt_1}(T-t_1)$$
$$-h\Bigg(a\mu_1e^{bt_1}+ae^{bt_1}\left(\mu_2-\mu_1-\frac{\theta}{2}(\mu_2-\mu_1)^2\right)$$
$$+a\left((t_1-\mu_2)+b({t_1}^2-\mu_2t_1)-\frac{\theta}{6}({t_1}^3-{\mu_2}^3)-\frac{b\theta}{6}({t_1}^4-{\mu_2}^3t_1)\right)\Bigg)$$
$$-PI_e a\left((t_1-M)+b({t_1}^2-Mt_1)-\frac{\theta}{6}({t_1}^3-M^3)-\frac{b\theta}{6}({t_1}^4-M^3t_1)\right)\Bigg\}$$

$$\frac{\partial^2 TP_5}{\partial {t_1}^2} = \frac{1}{T}\Bigg\{P_d\theta abe^{bt_1}$$
$$-h\Bigg(ab\mu_1e^{bt_1}+abe^{bt_1}\left(\mu_2-\mu_1-\frac{\theta}{2}(\mu_2-\mu_1)^2\right)$$
$$+\left[a+b(2t_1-\mu_2)-\frac{\theta}{6}(3{t_1}^2)-\frac{b\theta}{6}(4{t_1}^3-{\mu_2}^3)\right]\Bigg)$$
$$-PI_e\left[a+b(2t_1-M)-\frac{\theta}{6}(3{t_1}^2)-\frac{b\theta}{6}(4{t_1}^3-M^3)\right]+c_1ae^{bt_1}(1+bt_1)\Bigg\}$$

$$\frac{\partial TP_6}{\partial T} = -\frac{1}{T^2}\Bigg\{\frac{sa}{b}\left(e^{bT}(1-bT)-1\right) + P_d\theta\frac{a}{b}\left(e^{bt_1}-1\right) - A - P\frac{a}{b}\left(e^{bT}(1-bT)-1\right)$$

$$-h\Bigg(\frac{a}{b^2}\left(b\mu_1 - e^{b\mu_1}+1\right) + \frac{a}{b}\left(e^{bt_1}-1\right)\mu_1$$

$$-\frac{a}{2(b+\theta)}(\mu_2-\mu_1)^2(\theta+b-b\theta\mu_1)$$

$$+\frac{a}{b}\left(e^{bt_1}-e^{b\mu_1}\right)\left(\mu_2-\mu_1-\frac{\theta}{2}(\mu_2-\mu_1)^2\right)$$

$$+a\Bigg(\frac{1}{2}(t_1-\mu_2)^2 + \frac{b}{6}(2t_1{}^3 - 3\,\mu_2 t_1{}^2 + \mu_2{}^3) - \frac{\theta}{24}(t_1{}^4 - 4t_1\mu_2{}^3 + 3\mu_2{}^4)$$

$$-\frac{b\theta}{60}\left(2t_1{}^5 - 5\,\mu_2{}^3 t_1{}^2 + 3\mu_2{}^5\right)\Bigg)\Bigg)$$

$$-c_1\frac{a}{b}\left(\left(\frac{1}{b}-t_1\right)e^{bt_1} + \frac{1}{b}(bT-1)e^{bT}\right)$$

$$-(s-0.2s)\theta\left(\frac{t_1{}^3}{6} + \frac{bt_1{}^4}{8} - \frac{\theta t_1{}^5}{40} - \frac{b\theta t_1{}^6}{48} - \frac{\mu_2{}^2}{2}\left(t_1 + \frac{bt_1{}^2}{2} - \frac{2\mu_2}{3}\right)\right.$$

$$\left.+\frac{\mu_2{}^4}{8}\left(b+\theta t_1 + \frac{bt_1{}^2}{2}\right) - \frac{\mu_2{}^5}{2}\left(\frac{1}{5}+\frac{b\mu_2}{12}\right)\right)\Bigg\}$$

$$\frac{\partial^2 TP_6}{\partial T^2} = \frac{2}{T^3}\Bigg\{\frac{sa}{b}\left(e^{bT}(1-bT)-1\right) + P_d\theta\frac{a}{b}\left(e^{bt_1}-1\right) - A - P\frac{a}{b}\left(e^{bT}(1-bT)-1\right)$$

$$-h\Bigg(\frac{a}{b^2}\left(b\mu_1 - e^{b\mu_1}+1\right) + \frac{a}{b}\left(e^{bt_1}-1\right)\mu_1$$

$$-\frac{a}{2(b+\theta)}(\mu_2-\mu_1)^2(\theta+b-b\theta\mu_1)$$

$$+\frac{a}{b}\left(e^{bt_1}-e^{b\mu_1}\right)\left(\mu_2-\mu_1-\frac{\theta}{2}(\mu_2-\mu_1)^2\right)$$

$$+a\Bigg(\frac{1}{2}(t_1-\mu_2)^2 + \frac{b}{6}(2t_1{}^3 - 3\,\mu_2 t_1{}^2 + \mu_2{}^3) - \frac{\theta}{24}(t_1{}^4 - 4t_1\mu_2{}^3 + 3\mu_2{}^4)$$

$$-\frac{b\theta}{60}\left(2t_1{}^5 - 5\,\mu_2{}^3 t_1{}^2 + 3\mu_2{}^5\right)\Bigg)\Bigg) - c_1\frac{a}{b}\left(\left(\frac{1}{b}-t_1\right)e^{bt_1} + \left(T-\frac{1}{b}\right)e^{bT}\right)$$

$$-(s-0.2s)\theta\left(\frac{t_1{}^3}{6} + \frac{bt_1{}^4}{8} - \frac{\theta t_1{}^5}{40} - \frac{b\theta t_1{}^6}{48} - \frac{\mu_2{}^2}{2}\left(t_1 + \frac{bt_1{}^2}{2} - \frac{2\mu_2}{3}\right)\right.$$

$$\left.+\frac{\mu_2{}^4}{8}\left(b+\theta t_1 + \frac{bt_1{}^2}{2}\right) - \frac{\mu_2{}^5}{2}\left(\frac{1}{5}+\frac{b\mu_2}{12}\right)\right)\Bigg\} + \frac{abe^{bT}}{T}(s-P) + \frac{c_1 a e^{bT}}{T}$$

$$\frac{\partial TP_6}{\partial t_1}=\frac{1}{T}\left\{P_d\theta ae^{bt_1}-c_1ae^{bt_1}(T-t_1)\right.$$
$$-h\left(a\mu_1e^{bt_1}+ae^{bt_1}\left(\mu_2-\mu_1-\frac{\theta}{2}(\mu_2-\mu_1)^2\right)\right.$$
$$\left.+a\left((t_1-\mu_2)+b(t_1^2-\mu_2t_1)-\frac{\theta}{6}(t_1^3-\mu_2^3)-\frac{b\theta}{6}(t_1^4-\mu_2^3t_1)\right)\right)$$
$$-\theta(s-0.2s)\left(\frac{t_1^2}{2}+\frac{bt_1^3}{2}-\frac{\theta t_1^4}{8}-\frac{b\theta t_1^5}{8}-\frac{\mu_2^2}{2}+\frac{bt_1\mu_2^2}{2}\right.$$
$$\left.\left.+\frac{\theta\mu_2^4}{8}(1+bt_1)\right)\right\}$$

$$\frac{\partial^2 TP_6}{\partial t_1^2}=\frac{1}{T}\left\{P_d\theta abe^{bt_1}+c_1abe^{bt_1}(1+bt_1)\right.$$
$$-h\left(ab\mu_1e^{bt_1}+abe^{bt_1}\left(\mu_2-\mu_1-\frac{\theta}{2}(\mu_2-\mu_1)^2\right)\right.$$
$$\left.+\left[a+b(2t_1-\mu_2)-\frac{\theta}{6}(3t_1^2)-\frac{b\theta}{6}(4t_1^3-\mu_2^3)\right]\right)$$
$$\left.-(s-0.2s)\theta\left(t_1+\frac{3bt_1^2}{2}-\frac{\theta t_1^3}{2}-\frac{5b\theta t_1^4}{8}+\frac{b\mu_2^2}{2}+\frac{\theta b\mu_2^4}{8}\right)\right\}$$

Provided that above equations satisfies the following conditions

$$\frac{\partial^2 TP_i}{\partial T^2}<0,\frac{\partial^2 TP_i}{\partial t_1^2}<0 \qquad i=1,2,3,4,5,6 \qquad -----(43)$$

$$\left(\frac{\partial^2 TP_i}{\partial T^2}\right)\left(\frac{\partial^2 TP_i}{\partial t_1^2}\right)-\left(\frac{\partial^2 TP_i}{\partial T\partial t_1}\right)^2<0 \qquad i=1,2,3,4,5,6 \qquad -----(44)$$

4. Numerical examples

Case: I

Example 1:

Let A=100, a=250, b=0.1, P=22, s=35, h=3.5, c_1=3, P_d=12, θ=0.01, I_e=0.11, I_P=0.12, M=0.16, in appropriate units. The optimal value of Q_1=43(42.8633), t_1=0.17(0.1727), T=0.33(0.3263) and TP_1=3478(3478.14) per cycle.

Example 2:

Let A=100, a=250, b=0.1, P=22, s=35, h=3.5, c_1=3, P_d=12, θ=0.01, I_e=0.11, I_P=0.12, M=0.30, in appropriate units. The optimal value of Q_1=73(73.5615), t_1=0.29(0.2931), T=0.59(0.5929) and TP_1=3685(3685.97) per cycle.

Case: II

Example 3:

Let A=100, a=250, b=0.1, P=22, s=35, h=3.5, c_1=3, P_d=12, θ=0.01, I_e=0.11, I_P=0.12, M=0.08, μ_1=0.11, μ_2=0.16 in appropriate units. The optimal value of Q_1=45(45.074), t_1=0.18(0.1763), T=0.23(0.2348) and TP_1=3114(3113.54) per cycle.

Example 4:

Let A=100, a=250, b=0.1, P=22, s=35, h=3.5, c_1=3, P_d=12, θ=0.01, I_e=0.11, I_P=0.12, M=0.14, μ_1=0.11, μ_2=0.16 in appropriate units. The optimal value of Q_1=39(38.7824), t_1=0.15(0.1513), T=0.18(0.1831) and TP_1=3634(3634.07) per cycle.

Example 5:

Let A=100, a=250, b=0.1, P=22, s=35, h=3.5, c_1=3, P_d=12, θ=0.01, I_e=0.11, I_P=0.12, M=0.20, μ_1=0.11, μ_2=0.16 in appropriate units. The optimal value of Q_1=61(60.7258), t_1=0.24(0.2393), T=0.30(0.3023) and TP_1=3574(3574.04) per cycle.

Example 6:

Let A=100, a=250, b=0.1, P=22, s=35, h=3.5, c_1=3, P_d=12, θ=0.01, I_e=0.11, I_P=0.12, μ_1=0.11, μ_2=0.16 in appropriate units. The optimal value of Q_1=61(60.7258), t_1=0.24(0.2401), T=0.43(0.4299) and TP_1=3770(3769.93) per cycle.

5. Conclusion

In this paper, we have developed a purchasing, inventory model to determine the optimal cycle time and ordering quantity. Here two cases of the purchasing process in a cycle time are discussed. One is the ordering quantity with conforming quality items and the second one is theordering quantity with deteriorating items.This model is developed for deteriorating items with different deterioration rate under permissible delay in payments and price discount. In this model shortages are allowed and the demand was taken as anexponential function of time. Finally the proposed model was verified by the numerical example.

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