International Journal of Management, IT & Engineering
Vol. 8 Issue 1, January 2018,
ISSN: 2249-0558 Impact Factor: 7.119
Journal Homepage: http://www.ijmra.us, Email: editorijmie@gmail.com Double-Blind Peer Reviewed Refereed Open Access International Journal - Included in the International Serial Directories Indexed & Listed at: Ulrich's Periodicals Directory ©, U.S.A., Open J-Gate as well as in Cabell's Directories of Publishing Opportunities, U.S.A

ON SIX SELF-MAPS IN PARTIALLY ORDERED METRICSPACES

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Abstract: In this paper the existence of coincidence and common fixed point theorems for six self-maps satisfying contraction type in complete partially ordered metric spaceshave been proved.Our work is generalizations of earlier work of Sharmaet al. [22] and some others.

Keywords:partially ordered metric spaces, common fixed point, compatibility, weakly compatibility maps,weakly annihilator maps, dominating maps.

AMS Subject classification: Primary 54H25, Secondary 47H10.

1. Introduction

Jungck [11]introduced commuting maps, afterwards weakly commutativity, compatibility, compatibility of type (A), (B) and (P), weakly compatibility of maps have been established.(see [21, 13, 15, 16, 17, 14]etc).

Geraghty [11]generalized the Banach contraction principle.Afterward,Harandiet al. [6] extend the result of[11] in the context of partially ordered set.In [5]Altunet al. introduced weakly increasing maps.Further, Aydi [7] presented coincidence and common fixed point theorem for three weakly increasing self-maps.Later Al-Muhiameedet al.[3]extended the results ofAydi[7] for four maps by introducing and using the notions of weakly increasing, partially weakly increasing, weak annihilator, dominating, compatibility, weak compatibility of maps in partial ordered metric space.(see [2, 6, 8, 9, 18, 20] etc).

Recently, Sharma et al. [22]generalized the results of ([3], [7] and some others) forsome common fixed point theorems of four self-maps satisfying contraction type conditionwith an relevant example in partially ordered complete metric spaces.

In this paper, we generalized theresult of Sharma et al. [22] and some others for six self-mapsin the context of partially ordered complete metric spaces.

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2. Preliminaries

***Definition 2.1.*[5, 1]**Let $(X, \leq)$ be a partially ordered set.An ordered pair (f, g) of self maps of X is said to be

(a) weakly increasing if $fx \leq gfx$ and $gx \leq fgx$ for all $x \in X$.

(b) partially weakly increasing if $fx \leq gfx$ for all $x \in X$.

***Remark 2.2.* [1, 7]** (a) A pair (f, g) of self-maps of X is weakly increasing if and only if pair (f, g) and (g, f) are partially weakly increasing.

(b) A pair (f, g) of self-maps of X is weakly increasing$\Rightarrow$ the pair (f, g) is partially weakly increasing but the convers is not true.

***Definition 2.3.*[3, 1]**Let $(X, \leq)$ be a partially ordered set.

(a) A map f is called weak annihilator of g if $fgx \leq x$ for all $x \in X$.

(b) A map f is called dominating if $x \leq fx$ for all $x \in X$.

***Definition 2.4.*[13, 14]**Let (X, d) be a metric spaces.

(a) $f, g: X \to X$ are said to be compatible iff$\lim_{n\to\infty} d(fgx_n, gfx_n) = 0$ whenever $\{x_n\}$ is a sequence in X such that$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n =$ u for some $u \in X$.

(b) $f, g: X \to X$ are said to be weakly compatible if they commute at their coincidence points, that is, if $fx = gx$ for some $x \in X$ then$fgx = gfx$.

If S is the family of functions$\omega: R^+ \to [0, 1)$such that $\omega(t_n) \to 1$ implies $t_n \to 0$;

***Theorem 2.5.* [11]**Let $f: X \to X$ be a contraction of a complete metric space X satisfying

$(A1)$ $\quad d\big(f(x), f(y)\big) \leq \omega\big(d(x, y)\big) d(x, y), \quad \forall\, x,\ y \in X$, where$\omega \in S$ which need not be continuous. Then for any arbitrary point x_0 the iteration $x_n = f(x_{n-1})$, $n \geq 1$ converges to a unique fixed point of f in X.

Further, (Theorem 2.5) extended byHarandiet al. [6]

***Theorem 2.6.* [6]**Let $(X, \leq)$ be a partially ordered set and let there exists a metric d in X such that (X, d) is a complete metric space. Let $f: X \to X$ be a non-decreasing mapping such that there exists $x_0 \in X$ with$x_0 \leq f(x_0)$ satisfying (A1) for all$x,\ y \in X$ with$x \leq y$,

$(B1)$ either f is continuous or there exists a non-decreasing sequence$\{x_n\}$in X such that $x_n \to x$ then$x_n \leq x, \forall\, n$.

$(B2)$ for any $x,\ y \in X$, there exists $u \in X$ which is comparable to x and y.

Then f has a unique fixed point.

Further,Aydi, [7] proved the following result

***Theorem 2.7.*[7]**Let $(X,\leq)$ be a partially ordered set and let there exists a metric d in X such that (X,d) is a complete metric space. Let $f,\ g,\ H\colon X\to X$ are continuous mapping such that:

$(C1)$ $fX\subseteq HX,\ gX\subseteq HX.$

$(C2)\forall\ x,\ y\in X,\ Hx$ and Hy are comparable such that

$$d(fx,gy)\leq\omega\big(d(Hx,Hy)\big)d(Hx,Hy),\text{ where }\omega\in S$$

$(C3)$the pair (f,H) and (g,H) are compatible.

$(C4)f$ and g are weakly increasing with respect to H then f,g and H have a coincidence point. Moreover, if

$(C5)$for any $x,\ y\in X$ there exists $u\in X$ such that $fx\leq fu,\ fy\leq fu$ then f,g and H have a unique common fixed point.

Later ,Al-Muhiameedet al. [3]extended(Theorem 2.7)for four maps

***Theorem 2.8.*[3]** Let $(X,\leq)$ be a partially ordered set andlet there exists a metric d in X such that (X,d) is a complete metric space. Let $f,\ g,\ S,\ T\colon X\to X$ such that:

$(D1)$ $fX\subseteq TX$ and $gX\subseteq SX.$

$(D2)$for every comparable elements $x,\ y\in X,$

$d(fx,gy)\leq\omega\big(d(Sx,Ty)\big)d(Sx,Ty)$where$\omega\in S.$

$(D3)$ the pairs(T,f) and (S,g) are partially weakly increasing.

$(D4)f$ and g are dominating maps and weak annihilators of T and S, respectively.

$(D5)$there exists a non-decreasing sequence $\{x_n\}$ with $x_n\leq y_n$ for all n and $y_n\to u$ implies that$x_n\leq u$.

$(D6)$either pair (f,S) is compatible, pair (g,T) is weakly compatible and f or S is continuous map. Or

pair (g,T) is compatible, pair (f,S) is weakly compatible and g or T is continuous map.

Then $f,\ g,\ S$ and T have a common fixed point. Moreover, the set of common fixed points of $f,\ g,\ S$ and T is well ordered if and only if $f,\ g,\ S$ and T have a unique common fixed point.

Recently,(Theorem 2.8) is generalized bySharma et al. [22] by replacing $(D2)$ to

$(D7)d(fx,gy)\leq\omega\big(m(x,y)\big)m(x,y)$, where

$$m(x,y)=max\{\,d(Sx,Ty),d(fx,Sx),d(gy,Ty),\frac{1}{2}(d(Sx,gy)+d(fx,Ty))\}$$

for all $x,\ y \in X$ with $x \leq y$ and $\omega \in S$.

3. Main Result

Our main result have the following common fixed point theorem for six self-maps.

Theorem 3.1. Let $(X, \leq)$ be a partially ordered set andlet there exists a metric d in X such that (X,d) is a complete metric space. Let $\mathcal{A},\ \mathcal{B},\ \mathcal{F},\ \mathcal{G},\ S,\ V: X \to X$ satisfying (D5) and

$(E1)$ $\mathcal{AB}X \subseteq VX$ and $\mathcal{FG}X \subseteq SX$.

$(E2)$ $d(\mathcal{AB}x, \mathcal{FG}y) \leq \omega\big(m(x,y)\big)m(x,y)$, where

$$m(x,y) = max\left\{d(Sx,Vy), d(\mathcal{AB}x,Sx), d(\mathcal{FG}y,Vy), \frac{d(Sx,\mathcal{FG}y) + d(\mathcal{AB}x,Vy)}{2}\right\}$$

for all $x,\ y \in X$ with $x \leq y$ and $\omega \in S$.

$(E3)$(a) the pairs $(V, \mathcal{AB})$ and $(S, \mathcal{FG})$ are partially weakly increasing.

(b) $\mathcal{AB}$ and $\mathcal{FG}$ are dominating, and weak annihilators of V and S, respectively.

$(E4)$ one of $(\mathcal{AB})X, (\mathcal{FG})X, SX$ and VX is a complete subspace of X, then

(a) $\mathcal{FG}$ and V have a coincidence point in X,

(b) $\mathcal{AB}$ and S have a coincidence point in X.

$(E5)$ pairs $(\mathcal{AB}, S)$ and $(\mathcal{FG}, V)$ are weakly compatible then

(c) $\mathcal{AB},\ \mathcal{FG},\ S$ and V have a unique common fixed point in X.

Proof. Let x_0 be an arbitrary point in X, since $(\mathcal{AB})X \subseteq VX$ then there exists $x_1 \in X$ such that $(\mathcal{AB})x_0 = Vx_1$. Also since $(\mathcal{FG})X \subseteq SX$ then there exists $x_2 \in X$ such that $(\mathcal{FG})x_1 = Sx_2$. Inductivelywe can construct the sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$y_{2n} = \mathcal{AB}x_{2n} = Vx_{2n+1}$ and $y_{2n+1} = \mathcal{FG}x_{2n+1} = Sx_{2n+2}$ for all $n = 0, 1, 2, 3 \ldots$.

From $(E3)$, we have

$x_{2n} \leq \mathcal{AB}x_{2n} = Vx_{2n+1} \leq (\mathcal{AB})Vx_{2n+1} \leq x_{2n+1}$ and

$x_{2n+1} \leq \mathcal{FG}x_{2n+1} = Sx_{2n+2} \leq (\mathcal{FG})Sx_{2n+2} \leq x_{2n+2}$.

Thus $\forall\ n \geq 0$, we obtain $x_0 \leq x_1 \leq x_2 \leq x_3 \leq \cdots \leq x_n \leq x_{n+1} \ldots$.

Now we claim that $\{y_n\}$ is a Cauchy sequence in X. If $y_{2n} = y_{2n+1}$, for some n, then from $(E2)$, we have

$$d(y_{2n+1}, y_{2n+2}) = d(\mathcal{AB}x_{2n+2}, \mathcal{FG}x_{2n+1}) \leq \omega(m(x_{2n+2}, x_{2n+1}))m(x_{2n+2}, x_{2n+1}),$$

where

$$m(x_{2n+2}, x_{2n+1}) = max\{d(Sx_{2n+2}, Vx_{2n+1}), d(\mathcal{AB}x_{2n+2}, Sx_{2n+2}), d(\mathcal{FG}x_{2n+1}, Vx_{2n+1}), \frac{d(Sx_{2n+2}, \mathcal{FG}x_{2n+1}) + d(\mathcal{AB}x_{2n+2}, Vx_{2n+1})}{2}\}$$

$$= max\{d(y_{2n+1}, y_{2n}), d(y_{2n+2}, y_{2n+1}), d(y_{2n+1}, y_{2n+1}), \frac{d(y_{2n+1}, y_{2n+1}) + d(y_{2n+2}, y_{2n})}{2}\}$$

$$\leq max\{0, d(y_{2n+2}, y_{2n+1}), 0, \frac{d(y_{2n+2}, y_{2n+1}) + d(y_{2n+1}, y_{2n})}{2}\}$$

$$= max\{0, d(y_{2n+2}, y_{2n+1}), 0, \frac{d(y_{2n+2}, y_{2n+1}) + 0}{2}\} = d(y_{2n+1}, y_{2n+2}).$$

Hence, $d(y_{2n+1}, y_{2n+2}) \leq \omega\big(d(y_{2n+1}, y_{2n+2})\big)d(y_{2n+1}, y_{2n+2})$

Since $0 \leq \omega < 1$,we deduce that $d(y_{2n+1}, y_{2n+2}) \leq d(y_{2n+1}, y_{2n+2})$which is a contradiction. Hence we must have$y_{2n+1} = y_{2n+2}$using similar process, we obtain $y_{2n+2} = y_{2n+3}$ and so on. Thus$\{y_n\}$turns out to be a constant sequence and y_{2n} is the common fixed point of$\mathcal{AB}$, $\mathcal{FG}$, S and V.

Now supposed$d(y_{2n}, y_{2n+1}) > 0$ for every$n$, since $x = x_{2n}$ and$y = x_{2n+1}$ are comparable elements so using ($E2$) we obtain,

$d(y_{2n}, y_{2n+1}) = d(\mathcal{AB}x_{2n}, \mathcal{FG}x_{2n+1}) \leq \omega\big(m(x_{2n}, x_{2n+1})\big)m(x_{2n}, x_{2n+1})$…. (1)

where

$$m(x_{2n}, x_{2n+1}) = max\{d(Sx_{2n}, Vx_{2n+1}), d(\mathcal{AB}x_{2n}, Sx_{2n}), d(\mathcal{FG}x_{2n+1}, Vx_{2n+1}), \frac{d(Sx_{2n}, \mathcal{FG}x_{2n+1}) + d(\mathcal{AB}x_{2n}, Vx_{2n+1})}{2}\}$$

$$= max\{d(y_{2n-1}, y_{2n}), d(y_{2n}, y_{2n-1}), d(y_{2n+1}, y_{2n}), \frac{d(y_{2n-1}, y_{2n+1}) + d(y_{2n}, y_{2n})}{2}\}$$

$$\leq max\{d(y_{2n-1}, y_{2n}), d(y_{2n+1}, y_{2n}), \frac{d(y_{2n-1}, y_{2n}) + d(y_{2n}, y_{2n+1})}{2}\}$$

$$= max\{d(y_{2n-1}, y_{2n}), d(y_{2n+1}, y_{2n})\}$$

now

$m(x_{2n}, x_{2n+1}) =$ either$d(y_{2n+1}, y_{2n+2})$ or $d(y_{2n}, y_{2n+1})$

If$m(x_{2n}, x_{2n+1}) = d(y_{2n+1}, y_{2n+2})$then from (1) we have

$$d(y_{2n+1}, y_{2n+2}) = d(\mathcal{AB}x_{2n}, \mathcal{FG}x_{2n+1}) \leq \omega\big(d(y_{2n+2}, y_{2n+1})\big)d(y_{2n+2}, y_{2n+1})$$

Since$0 \leq \omega < 1$, we deduce that $d(y_{2n+1}, y_{2n+2}) \leq d(y_{2n+1}, y_{2n+2})$ which is a contradiction.

Therefore $m(x_{2n}, x_{2n+1}) = d(y_{2n}, y_{2n+1})$hencefrom (1) we obtain,

$d(y_{2n+1}, y_{2n+2}) = d(\mathcal{AB}x_{2n}, \mathcal{FG}x_{2n+1}) \leq \omega\big(d(y_{2n}, y_{2n+1})\big)d(y_{2n}, y_{2n+1})$ …. (2)

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Since$0 \leq \omega < 1$, we deduce that $d(y_{2n+1}, y_{2n+2}) \leq d(y_{2n}, y_{2n+1})$

By using similar arguments for $x = x_{2n-1}$ and $y = x_{2n}$ in $(E2)$ we have

$d(y_{2n}, y_{2n+1}) \leq d(y_{2n-1}, y_{2n})$

Hence for any $n, d(y_{n+2}, y_{n+1}) \leq d(y_{n+1}, y_n) \leq d(y_n, y_{n-1}) \leq \cdots \leq d(y_1, y_0)$ implies that the sequence $\{d(y_{n+1}, y_n)\}$ is monotonicnon-increasing sequence.

Hence there exists $r \geq 0$ such that $\lim_{n\to\infty} d(y_{n+1}, y_n) = r$…. (3)

Using (2) we have,

$$\frac{d(y_{2n+1}, y_{2n+2})}{d(y_{2n}, y_{2n+1})} \leq \omega\big(d(y_{2n}, y_{2n+1})\big) < 1$$

Letting$n \to \infty$ in the above inequality, then from (3) we obtain

$\lim_{n\to\infty} \omega\big(d(y_{2n}, y_{2n+1})\big) = 1$and since $\omega \in S$ this implies that$r = 0$.

Hence for any $n, \lim_{n\to\infty} d(y_{n+1}, y_n) = 0$ ….(4)

Now we claim that $\{y_n\}$ is a Cauchy sequence.Suppose on the contrary that $\{y_{2n}\}$ is not a Cauchy sequence then there is$\varepsilon > 0$, and there exist even integers $2m_k$, $2n_k$with $2m_k > 2n_k > k$ for all $k > 0$ such that

$d(y_{2m_k}, y_{2n_k}) \geq \epsilon$, …. (5)and$d(y_{2m_k-2}, y_{2n_k}) < \epsilon$….(6)

Now using (5), (6) and by triangle inequality, we have

$$\epsilon \leq d(y_{2m_k}, y_{2n_k}) \leq d(y_{2n_k}, y_{2m_k-2}) + d(y_{2m_k-1}, y_{2m_k-2}) + d(y_{2m_k-1}, y_{2m_k})$$
$$\leq \epsilon + d(y_{2m_k-1}, y_{2m_k-2}) + d(y_{2m_k-1}, y_{2m_k})$$

Letting$k \to \infty$ in the above inequality and using (4), we obtain

$\lim_{k\to\infty} d(y_{2m_k}, y_{2n_k}) = \epsilon$…. (7)

Again for all $k > 0$, (4) and inequality

$$d(y_{2m_k}, y_{2n_k}) \leq d(y_{2m_k}, y_{2m_k-1}) + d(y_{2m_k-1}, y_{2n_k}) \Rightarrow \epsilon \leq \lim_{k\to\infty} d(y_{2m_k-1}, y_{2n_k})$$

While (4) and inequality

$$d(y_{2m_k-1}, y_{2n_k}) \leq d(y_{2m_k-1}, y_{2m_k}) + d(y_{2m_k}, y_{2n_k}) \Rightarrow \lim_{k\to\infty} d(y_{2m_k-1}, y_{2n_k}) \leq \epsilon$$

Hence $\lim_{k\to\infty} d(y_{2m_k-1}, y_{2n_k}) = \epsilon$…. (8)

Again for all $k > 0$, (4) and inequality

$$d(y_{2m_k}, y_{2n_k}) \leq d(y_{2m_k}, y_{2n_k+1}) + d(y_{2n_k+1}, y_{2n_k}) \Rightarrow \epsilon \leq \lim_{k\to\infty} d(y_{2m_k}, y_{2n_k+1})$$

while (4) and inequality

$$d(y_{2m_k}, y_{2n_k+1}) \leq d(y_{2n_k}, y_{2n_k+1}) + d(y_{2m_k}, y_{2n_k}) \Rightarrow \lim_{k\to\infty} d(y_{2m_k-1}, y_{2n_k}) \leq \epsilon$$

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Hence $\lim_{k\to\infty} d(y_{2m_k}, y_{2n_k+1}) = \epsilon$.... (9)

Now taking $x = x_{2n_k}$ and $y = x_{2m_k-1}$ in the contractive condition $(E2)\forall\ k > 0$, we have

$$d(y_{2n_k+1}, y_{2m_k}) = d(\mathcal{AB}x_{2n_k}, \mathcal{FG}x_{2m_k-1})$$

$$\leq \omega\left(m(x_{2n_k}, x_{2m_k-1})\right) m(x_{2n_k}, x_{2m_k-1})\ldots.(10)$$

where

$$m(x_{2n_k}, x_{2m_k-1})$$

$$= \max\{d(Sx_{2n_k}, Vx_{2m_k-1}), d(\mathcal{AB}x_{2n_k}, Sx_{2n_k}), d(\mathcal{FG}x_{2m_k-1}, Vx_{2m_k-1}),$$

$$\frac{1}{2}(d(Sx_{2n_k}, \mathcal{FG}x_{2m_k-1}) + d(\mathcal{AB}x_{2n_k}, Vx_{2m_k-1}))\}$$

$$= max\{ d(y_{2n_k}, y_{2m_k-1}), d(y_{2n_k+1}, y_{2n_k}), d(y_{2m_k}, y_{2m_k-1}),$$

$$\frac{1}{2}(d(y_{2n_k}, y_{2m_k}) + d(y_{2n_k+1}, y_{2m_k-1}))\}$$

$$= max\{ d(y_{2n_k}, y_{2m_k-1}), d(y_{2n_k+1}, y_{2n_k}), d(y_{2m_k}, y_{2m_k-1}),$$

$$\frac{1}{2}(d(y_{2n_k}, y_{2m_k-1}) + d(y_{2m_k}, y_{2n_k+1}))\} \quad \ldots. (11)$$

Letting$k \to \infty$ in (11) and using (4), (7), (8) and (9), we have

Thus $\lim_{k\to\infty} m(x_{2n_k}, x_{2m_k-1}) = \max\left\{\epsilon, 0, 0, \frac{(\epsilon+\epsilon)}{2}\right\} = \epsilon$.... (12)

Therefore, since $y_{2n_k+1} \neq y_{2m_k}$ then$\frac{d(\mathcal{AB}x_{2n_k}, \mathcal{FG}x_{2m_k-1})}{m(x_{2n_k}, x_{2m_k-1})} < \omega\left(m(x_{2n_k}, x_{2m_k-1})\right) < 1$

Using the fact $\epsilon = \lim_{k\to\infty} d(\mathcal{AB}x_{2n_k}, \mathcal{FG}x_{2m_k-1}) = \lim_{k\to\infty} m(x_{2n_k}, x_{2m_k-1})$, we get

$\lim_{k\to\infty} \omega\left(m(x_{2n_k}, x_{2m_k-1})\right) = 1$ since$\omega \in S$,hence $\lim_{k\to\infty} m(x_{2n_k}, x_{2m_k-1}) = 0$ which is contradiction i.e. $\lim_{k\to\infty} m(x_{2n_k}, x_{2m_k-1}) = \epsilon > 0$.Thus $\{y_{2n}\}$ is a Cauchy sequence and since$X$ is complete so there exist a point z in X such that$\{y_n\}$ and its sub sequences $\{y_{2n+1}\}$ and $\{y_{2n}\}$ are also converges to$z$. i.e.,

$\lim_{n\to\infty} y_{2n} = \lim_{n\to\infty} \mathcal{AB}x_{2n} = \lim_{n\to\infty} V\, x_{2n+1} = z$ and

$\lim_{n\to\infty} y_{2n+1} = \lim_{n\to\infty} \mathcal{FG}\, x_{2n+1} = \lim_{n\to\infty} Sx_{2n+2} = z$

Suppose VX is closed then there exists $w^* \in X$ such that $z = Vw^*$. From $(E3)$, since $x_{2n} \leq \mathcal{AB}x_{2n}and\mathcal{AB}x_{2n} \to z$ as $n \to \infty \Rightarrow x_{2n} \leq w^* = Vw^* \leq (\mathcal{AB})Vw^* \leq w^*$.

Using $(E2)$, we have

$d(\mathcal{AB}x_{2n}, \mathcal{FG}w^*) \leq \omega(m(x_{2n}, w^*))m(x_{2n}, w^*)$....(13)

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where

$$m(x_{2n}, w^*) = max\{d(Sx_{2n}, Vw^*), d(\mathcal{AB}x_{2n}, Sx_{2n}), d(\mathcal{FG}w^*, Vw^*), \frac{d(Sx_{2n}, \mathcal{FG}w^*) + d(\mathcal{AB}x_{2n}, Vw^*)}{2}\}$$

$$= \max\{d(Sx_{2n}, z), d(\mathcal{AB}x_{2n}, Sx_{2n}), d(\mathcal{FG}w^*, z), \frac{d(Sx_{2n}, \mathcal{FG}w^*) + d(\mathcal{AB}x_{2n}, z)}{2}\}$$

Letting $n \to \infty$, we have

$$\lim_{n\to\infty} m(x_{2n}, w^*) = max\{ d(z,z), d(z,z), d(\mathcal{FG}w^*, z), \frac{1}{2}(d(z, \mathcal{FG}w^*) + d(z,z))\}$$

$$= max\{ 0, 0, d(\mathcal{FG}w^*, z), \frac{1}{2}(d(z, \mathcal{FG}w^*) + 0)\} = d(\mathcal{FG}w^*, z)$$

Hence from (13) as $n \to \infty$, we have

$d(z, \mathcal{FG}w^*) \le \omega\big(d(\mathcal{FG}w^*, z)\big)d(\mathcal{FG}w^*, z)$since$\omega \in S$ this implies that $\mathcal{FG}w^* = z$.

So $\mathcal{FG}w^* = Vw^* = z$.

Now by weakly compatibility of pair $(\mathcal{FG}, V)$, $\mathcal{FG}z = (\mathcal{FG})Vw^* = V(\mathcal{FG})w^* = Vz$.

Using ($E2$), we have

$d(z, \mathcal{FG}z) = d(\mathcal{AB}x_{2n}, \mathcal{FG}z) \le \omega(m(x_{2n}, z))m(x_{2n}, z)$….(14)

where

$$m(x_{2n}, z) = \max\{d(Sx_{2n}, Vz), d(\mathcal{AB}x_{2n}, Sx_{2n}), d(\mathcal{FG}z, Vz), \frac{d(Sx_{2n}, \mathcal{FG}z) + d(\mathcal{AB}x_{2n}, Vz)}{2}\}$$

$$= \max\{d(Sx_{2n}, Vz), d(\mathcal{AB}x_{2n}, Sx_{2n}), d(\mathcal{FG}z, Vz), \frac{d(Sx_{2n}, \mathcal{FG}z) + d(\mathcal{AB}x_{2n}, Vz)}{2}\}$$

$$= \max\{d(Sx_{2n}, \mathcal{FG}z), d(\mathcal{AB}x_{2n}, Sx_{2n}), d(\mathcal{FG}z, \mathcal{FG}z), \frac{d(Sx_{2n}, \mathcal{FG}z) + d(\mathcal{AB}x_{2n}, \mathcal{FG}z)}{2}\}$$

Letting $n \to \infty$, we have

$$\lim_{n\to\infty} m(x_{2n}, z) = max\{ d(z, \mathcal{FG}z), d(z,z), d(\mathcal{FG}z, \mathcal{FG}z), \frac{1}{2}(d(z, \mathcal{FG}z) + d(z, \mathcal{FG}z))\}$$

$$= max\{ d(z, \mathcal{FG}z), 0, 0, d(z, \mathcal{FG}z)\} = d(z, \mathcal{FG}z),$$

Therefore from (14) as $n \to \infty$, we have

$d(z, \mathcal{FG}z) \le \omega\big(d(z, \mathcal{FG}z)\big)d(z, \mathcal{FG}z)$, since$\omega \in S$ this implies that

Hence $\mathcal{FG}z = z$. …. (15)

Since $\mathcal{FG}X \subseteq SX$ then there exists a point $v^* \in X$ such that $z = \mathcal{FG}z = Sv^*$.

From $(E3)$ since $z \leq \mathcal{FG}z = Sv^* \leq (\mathcal{FG})Sv^* \leq v^*$ implies that $z \leq v^*$.

Using $(E2)$, we have

$d(\mathcal{AB}v^*, Sv^*) = d(\mathcal{AB}v^*, \mathcal{FG}z) \leq \omega(m(v^*, z))m(v^*, z)$…. (16)

where

$$m(v^*, z) = max\{d(Sv^*, Vz), d(\mathcal{AB}v^*, Sv^*), d(\mathcal{FG}z, Vz), \frac{d(Sv^*, \mathcal{FG}z) + d(\mathcal{AB}v^*, Vz)}{2}\}$$

$$= max\{d(Sv^*, Vz), d(\mathcal{AB}v^*, Sv^*), d(\mathcal{FG}z, Vz), \frac{d(Sv^*, \mathcal{FG}z) + d(\mathcal{AB}v^*, \mathcal{FG}z)}{2}\}$$

$$= max\{d(\mathcal{FG}z, Vz), d(\mathcal{AB}v^*, Sv^*), d(\mathcal{FG}z, Vz), \frac{d(Sv^*, \mathcal{FG}z) + d(\mathcal{AB}v^*, Sv^*)}{2}\}$$

$$= max\{d(z, z), d(\mathcal{AB}v^*, Sv^*), d(z, z), \frac{d(z, z) + d(\mathcal{AB}v^*, Sv^*)}{2}\}$$

$$= max\{0, d(\mathcal{AB}v^*, Sv^*), 0, \frac{0 + d(\mathcal{AB}v^*, Sv^*)}{2}\} = d(\mathcal{AB}v^*, Sv^*)$$

Therefore from (16), we have

$d(\mathcal{AB}v^*, Sv^*) \leq \omega\big(d(\mathcal{AB}v^*, Sv^*)\big)d(\mathcal{AB}v^*, Sv^*)$ since $\omega \in S$ this implies that $\mathcal{AB}v^* = Sv^*$. Now by weakly compatibility of pair $(\mathcal{AB}, S)$, $\mathcal{AB}z = (\mathcal{AB})Sv^* = S(\mathcal{AB})v^* = Sz$.

Using $(E2)$, we have

$d(\mathcal{AB}z, z) = d(\mathcal{AB}z, \mathcal{FG}z) \leq \omega(m(z, z))m(z, z)$…. (17)

where

$$m(z, z) = max\{d(Sz, Vz), d(\mathcal{AB}z, Sz), d(\mathcal{FG}z, Vz), \frac{d(Sz, \mathcal{FG}z) + d(\mathcal{AB}z, Vz)}{2}\}$$

$$= max\{d(\mathcal{AB}z, Vz), d(Sz, Sz), d(z, z), \frac{d(\mathcal{AB}z, z) + d(\mathcal{AB}z, z)}{2}\}$$

$$= max\{d(\mathcal{AB}z, z), 0, 0, d(\mathcal{AB}z, z)\} = d(\mathcal{AB}z, z)$$

Therefore from (17), we have

$d(\mathcal{AB}z, z) \leq \omega(d(\mathcal{AB}z, z))d(\mathcal{AB}z, z)$, since $\omega \in S$ this implies that

$\mathcal{AB}z = Sz = z$ …. (18)

Hence from (15) and (18), we have $\mathcal{AB}z = \mathcal{FG}z = Sz = Vz = z$, i.e. z is the common fixed point of $\mathcal{AB}$, $\mathcal{FG}$, S and V.

For the uniqueness of z suppose u^* be another common fixed point of $\mathcal{AB}$, $\mathcal{FG}$, S and V then from ($E2$), we have

$$d(z,u^*) = d(\mathcal{AB}z, \mathcal{FG}u^*) \leq \omega\big(m(z,u^*)\big)m(z,u^*)\ldots.(19)$$

where

$$m(z,u^*) = max\{d(Sz,Vu^*), d(\mathcal{AB}z,Sz), d(\mathcal{FG}u^*,Vu^*), \frac{d(Sz,\mathcal{FG}u^*) + d(\mathcal{AB}z,Vu^*)}{2}\}$$

$$= max\{d(z,u^*), 0{,}0, d(z,u^*)\} = d(z,u^*)$$

Therefore from (19), we have

$d(z,u^*) \leq \omega\big(d(z,u^*)\big)d(z,u^*)$, since $\omega \in S$ this implies that $z = u^*$, i.e. z is the unique common fixed point of $\mathcal{AB}$, $\mathcal{FG}$, S and V.

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