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SUM AND PRODUCT THEOREMS ON RELATIVE ORDER, RELATIVE TYPE AND WEAK TYPE OF BI-COMPLEX ENTIRE FUNCTIONS

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ABSTRACT

Orders and types of entire functions have already been investigated by many authors in the field of advanced complex analysis. Sum and product theorems on order and type of entire functions have been established and extended in many ways. In this paper we wish to establish some of the basic properties over sum and product theorems on order (lower order), type (lower type) and weak type of bi-complex entire functions in the field of advanced bi-complex analysis.

KEYWORDS:

Entire function; Relative order; Relative type; Sum and product theorems; Regular relative growth; Property (A).

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1. INTRODUCTION

After the introduction of Bi-complex number by the eminent mathematician C. Segre [15] in 1892, several research works have been done and a lot of improvement has been incorporated by the interested researchers during the last few decades. Some of the renowned mathematicians ([5], [6], [7], [8], [18]) made successful investigations on the properties of relative growth properties of entire functions. Growth properties of entire functions related to order, lower order, type, lower type, etc. are widely discussed in the study of advanced complex analysis since last few years. Several results have been established and extended related to sum and product theorems on growth properties of entire functions also. In this paper we wish to investigate some basic properties of relative order, relative lower order, relative type, relative lower type and relative weak type of bi-complex entire functions under some different conditions (with respect to sum and product of bi-complex entire functions). We do not need to explain the standard definitions and notations of the theory of bi-complex entire functions as those are available in [13].

2. DEFINITIONS AND NOTATIONS

We use some useful definitions and notations as mentioned below in the field of bi-complex entire functions.

Definition.2.1 [15] Bi-complex number

A bi-complex number is defined as $T=\{z_1+i_2z_2 / z_1,z_2 \in C(i_1)\}$, where the imaginary units i_1, i_2 follow the rules $i_1^2=i_2^2=-1, i_1i_2=i_2i_1=j$, say and $j^2=1$ *etc.*

Another representation is: $T=\{w_0+w_1i_1+w_2i_2+w_3j / w_i \in R, i=0,1,2,3\}$

Definition.2.2. Idempotent representation of a bi-complex number

Every bi-complex number $(z_1+i_2z_2)$ has the following idempotent representation:

$z_1+i_2z_2=(z_1-i_1z_2)e_1+(z_1+i_1z_2)e_2$, where $e_1=\frac{1+i_1i_2}{2}, e_2=\frac{1-i_1i_2}{2}$.

Definition.2.3. Bi-complex Entire functions

Let U be an open set of T and $w_0 \in U$. Then $f: U \subseteq T \to T$ is said to be entire in U if

$f'(w_0) \in T$ for all $w_0 \in U$, where $\lim_{w \to w_0} \frac{f(w)-f(w_0)}{w-w_0} = f'(w_0)$.

Definition.2.4. Idempotent representation of a bi-complex entire function

Let X1, X_2 be open sets in $C(i_1)$ and $T \subset C(i_2)$. Then a bi-complex function $f(z_1+i_2z_2)=f_{e_1}(z_1-i_1z_2)e_1+f_{e_2}(z_1+i_1z_2)e_2$ for all $z_1+i_2z_2 \in X_1 \times_e X_2$, is said to be entire if and only if $f_{e_1}: X_1 \to C(i_1)$ and $f_{e_2}: X_2 \to C(i_1)$ are entire functions and $f'(z_1+i_2z_2)=f'_{e_1}(z_1-i_1z_2)e_1+f'_{e_2}(z_1+i_1z_2)e_2$.

Definition.2.5[13]. Pole of a bi-complex function.

Let f : X → T be a bi-complex meromorphic function on the open set $X \subset T$. We can say that $w = (z_1 - i_1 z_2)e_1 + (z_1 + i_1 z_2)e_2 \in X$ is a pole for the bi-complex meromorphic function $f(w) = f(z_1 + i_2 z_2) = f_{e_1}(z_1 - i_1 z_2)e_1 + f_{e_2}(z_1 + i_1 z_2)e_2$,

if $z_1 - i_1 z_2 \in P_1(X)$ and $z_1 + i_1 z_2 \in P_2(X)$ are poles for $f_{e_1} : P_1(X) \to C(i_1)$ and $f_{e_2} : P_2(X) \to C(i_1)$ respectively.

Proposition.2.1

Let f : X → T be a bi-complex meromorphic function on the open set $X \subset T$. If $w_0 \in X$ then w_0 is a pole of f, if and only if $\lim_{w \to w_0} |f(w)| = \infty$.

Definition . 2.6 Order of a bi-complex function.

The order ρ (F) of a bi-complex entire function $F(w) = F_{e_1}(z_1 - i_1 z_2)e_1 + (z_1 + i_1 z_2)e_2$ is defined as $\rho(F) = Max\{\rho_{F_{e_1}}, \rho_{F_{e_2}}\}$

where $\rho_{F_{e_i}} = \limsup_{r_i \to \infty} \dfrac{\log\log M_i(r_i, F_{e_i})}{\log r_i}$ for i = 1,2.

Remark .2.1

The lower order λ(F) of a bi-complex entire function is defined as

$\lambda(F) = Min\{\lambda(F_{e_1}), \lambda(F_{e_2})\}$. where $\lambda_{F_{e_i}} = \liminf_{r_i \to \infty} \dfrac{\log\log M_i(r_i, F_{e_i})}{\log r_i}$ for i = 1,2.

Definition. 2.7. The type of F

The type σ(F) of a bi-complex entire function is defined as $\sigma(F) = Max\{\sigma(F_{e_1}), \sigma(F_{e_2})\}$ where $\sigma(F_{e_i}) = \limsup_{r_i \to \infty} \dfrac{\log M_i(r_i, F_{e_i})}{r_i^{\rho_{Fei}}}$ and $0 < \rho_{F_{e_i}} < \infty$ for *i=1,2.*

Definition. 2.8. The lower type of F

The lower type $\bar{\sigma}(F)$ of a bi-complex function is defined as $\bar{\sigma}(F) = Min\{\bar{\sigma}(F_{e_1}), \bar{\sigma}(F_{e_2})\}$ where $\bar{\sigma}(F_{e_i}) = \liminf_{rl \to \infty} \dfrac{\log M_i(r_i, F_{e_i})}{r_i^{\rho_{Fei}}}$ and $0 < \rho_{F_{e_i}} < \infty$

for *i=1,2.*

Definition. 2.9. The weak type of F

The weak type $\tau(F)$ of a bi-complex entire function is defined as $\tau(F) = Min\{\tau(F_{e_1}), \tau(F_{e_2})\}$ where $\tau(F_{e_i}) = \liminf_{rl \to \infty} \frac{\log M_i(r_i, F_{e_i})}{r_i^{\lambda_{F_{ei}}}}$ and $0 < \lambda_{F_{e_i}} < \infty$ for *i=1,2.*

Definition. 2.10 Maximum Modulus Function

Let *F(w)* be an bi-complex entire function defined in the open space C_2. The function $M_F(r)$ *on* $|w| = r$, known as the maximum modulus function corresponding to *F(w)* is defined as follows: $M_F(r) = max_F |w| = r|F(w)|$.

For any two given entire functions *F(w) and G(w)*, the ratio $\frac{M_F(r)}{M_G(r)}$ *as* $r \to \infty$ is called the growth of F with respect to G in terms of their maximum moduli.

Definition. 2.11. Relative order of G(w) w.r.t. F(w)

According to the definition of relative order of an entire function G with respect to an entire function F introduced by the renowned mathematician L. Bernal [2], the relative order of G(w) is defined as follows:

$$\rho_{F_{ei}}(G_{ei}) = \inf\{\mu > 0 : M_{G_{ei}}(r) < M_{F_{ei}}(r^{\mu}) for\ all\ r > r_0(\mu) > 0\} = \lim_{r\to\infty} \sup \frac{\log M_{F_{ei}}^{-1}(M_{G_{ei}}(r))}{\log r}, i = 1,2.$$

Therefore, $\rho_F(G) = Max\{\rho_{F_{ei}}(G_{ei}) : i = 1,2\}$.

It is quite definite that the above definition coincides with the classical one if *f(w) = Exp(w).*

Definition. 2.12. Relative lower order of f(z)

Similarly, the relative lower order of G with respect to F, denoted by $\lambda_F(G)$, can be defined as follows:

$$\lambda_F(G) = Min\{\lambda_{F_{ei}}(G_{ei}) : i = 1,2\}\ ,\ where\ \lambda_{F_{ei}}(G_{ei}) = \lim_{r\to\infty} \inf \frac{\log M_{F_{ei}}^{-1}(M_{G_{ei}}(r))}{\log r}.$$

Definition. 2.13. Regular relative growth

An entire function G is said to be of regular relative growth with respect to F if its relative order with respect to F coincides with its relative lower order with respect to F.

Definition. 2.14. [6] Property (A) of an entire function

A non-constant entire function F is said to have the Property (A) if for any $\sigma > 1$ and for all large values of r, $[M_F(r)]^2 \leq M_F(r^\sigma)$ holds. For further illustration and examples with or without the Property (A), one may see [3].

To compare the relative growth of two entire functions having same non-zero finite relative order with respect to another entire function, Roy [14] recently introduced the notion of relative type of two entire functions from which one can obtain the following definitions:

Definition. 2.15. Relative Type

Let F and G be two bi-complex entire functions such that $0 < \rho_G(F) < \infty$. Then the relative type $\sigma_G(F)$ of F with respect to G is defined as follows:

$$\sigma_G(F) = Max\{\sigma_{G_{ei}}(F_{ei}): i = 1,2\},$$

$$where\ \ \sigma_{G_{ei}}(F_{ei}) = \inf\{k > 0 :\ M_F(r) < M_G\left(kr^{\rho_G(F)}\right) for\ all\ sufficiently\ large\ value\ of\ r\} = \lim_{r\to\infty}\sup\frac{M_G^{-1}(M_F(r)}{r^{\rho_G(F)}}.$$

Definition. 2.16. Relative Lower Type

Let F and G be two bi-complex entire functions such that $0 < \rho_G(F) < \infty$. Then the relative lower type $\sigma_G(F)$ of F with respect to G is defined as follows:

$$\bar{\sigma}_G(F) = Min\{\sigma_{G_{ei}}(F_{ei}): i = 1,2\},\ \text{where}\ \ \sigma_{G_{ei}}(F_{ei}) = \lim_{r\to\infty}\inf\frac{M_{G_{ei}}^{-1}(M_{F_{ei}}(r)}{r^{\rho_{G_{ei}}(F_{ei})}}.$$

Analogously, to determine the relative growth of two entire functions having same non-zero finite relative lower order with respect to another entire function, Datta, Biswas & Sen [3] introduced 'relative weak type' of an entire function f with respect to another entire function g of finite positive relative lower order $\lambda_g(f)$. According to that, one can define the relative weak type of a bi-complex entire function in the following way:

Definition. 2.17. Relative Weak Type

Let F and G be two bi-complex entire functions such that $0 < \lambda_G(F) < \infty$. Then the relative weak type $\tau_G(F)$ of F with respect to G is defined as follows:

$$\tau_G(F) = Min\{\tau_{G_{ei}}(F_{ei}): i = 1,2\}, \text{ where } \tau_{G_{ei}}(F_{ei}) = \lim_{r\to\infty} \inf \frac{M_{G_{ei}}^{-1}(M_{F_{ei}}(r)}{r^{\lambda_{G_{ei}}(F_{ei})}}$$

If we choose, $G(W) = \text{Exp}(w)$, we may easily verify that the above definitions coincide with the classical definitions of type, lower type and weak type respectively.

3. LEMMAS

Analytical determination of order and type of entire functions are very much significant in the study of the basic growth properties in the value distribution theory. During the past few decades, many of the researchers have made close investigations on this subject area to yield many good results, for example, some of which may be recalled here.

Lemma. 3.1

If f(z) be an entire function and α *and* β be such that $\alpha > 1$ *and* $0 < \beta < \alpha$, then for all large value of r,

$$M_f(\alpha r) > \beta M_f(r)$$

Lemma. 3.2

Let f(z) and g(z) be any two entire functions of order ρ_f *and* ρ_g respectively. Then

$(i)\ \rho_{f+g} = \rho_g \ when\ \rho_f < \rho_g\ and\ (ii)\ \rho_{f.g} \le \rho_g\ when\ \rho_f \le \rho_g$ respectively.

Lemma. 3.3

Let f(z) and g(z) be any two entire functions of order ρ_f *and* ρ_g respectively. Then

$(i)\ \rho_{f+g} \le \max\{\rho_f, \rho_g\}\ and\ (ii)\ \rho_{f.g} \le \max\{\rho_f, \rho_g\}$ respectively.

Lemma.3.4

Let f(z) and g(z) be any two entire functions of type σ_f *and* σ_g respectively. Then

$(i)\ \sigma_{f+g} \le \sigma_g\ when\ \sigma_f < \sigma_g$, $(ii)\ \sigma_{f.g} \le \sigma_f + \sigma_g\ and\ (iii)\ \sigma_{f+g} \le \max\{\sigma_f, \sigma_g\}$ respectively.

Lemma. 3.5

Let f(z) and g(z) be any two entire functions of type σ_f *and* σ_g respectively. Then

$(i)\ \sigma_{f+g} \le \max\{\sigma_f, \sigma_g\}\ and\ (ii)\ \rho_{f.g} \le \sigma_f + \sigma_g$ respectively.

Lemma. 3.6

If $f_1(w)$, $g_1(w)$ and $g_2(w)$ be any three bi-complex entire functions and if $\rho_{f_1}(g_i) = \max\{\rho_{f_1}(g_k) \text{ for } k, i = 1,2\}$, then $\rho_{f_1}(g_1 \pm g_2) \leq \rho_{f_1}(g_i), \text{for } i = 1,2$; where the equality holds when $\rho_{f_1}(g_1) \neq \rho_{f_1}(g_2)$.

Lemma. 3.7

If $f_1(w)$, $g_1(w)$ and $g_2(w)$ be any three bi-complex entire functions and if $\rho_{f_1}(g_i) = \max\{\rho_{f_1}(g_k) \text{ for } k, i = 1,2\}$, then $\rho_{f_1}(g_1 . g_2) \leq \rho_{f_1}(g_i), \text{for } i = 1,2$; where the equality holds when $\rho_{f_1}(g_1) \neq \rho_{f_1}(g_2)$.

Lemma. 3.8

If $f_1(w)$, $g_1(w)$ and $g_2(w)$ be any three bi-complex entire functions such that at least one of g_1 and g_2 is of regular relative growth with respect to f_1 and $\lambda_{f_1}(g_i) = \max\{\lambda_{f_1}(g_k) \text{ for } k, i = 1,2\}$, then $\lambda_{f_1}(g_1 \pm g_2) \leq \lambda_{f_1}(g_i), \text{for } i = 1,2$; where the equality holds when $\lambda_{f_1}(g_1) \neq \lambda_{f_1}(g_2)$.

Lemma. 3.9

If $f_1(w)$, $g_1(w)$ and $g_2(w)$ be any three bi-complex entire functions such that f_1 has the Property(A) and at least one of g_1 and g_2 is of regular relative growth with respect to f_1 and $\lambda_{f_1}(g_i) = \max\{\lambda_{f_1}(g_k) \text{ for } k, i = 1,2\}$, then $\lambda_{f_1}(g_1 . g_2) \leq \lambda_{f_1}(g_i), \text{for } i = 1,2$; where the equality holds when $\lambda_{f_1}(g_1) \neq \lambda_{f_1}(g_2)$.

Lemma. 3.10

If $f_1(w)$, $g_1(w)$ and $g_2(w)$ be any three bi-complex entire functions such that (i) $\rho_{f_1}(g_i) = \max\{\rho_{f_1}(g_k) \text{ for } k, i = 1,2\}$ and (ii) $\rho_{f_1}(g_1) \neq \rho_{f_1}(g_2)$ hold, then $\sigma_{f_1}(g_1 \pm g_2) = \sigma_{f_1}(g_i), \text{for } i = 1,2.$

Lemma. 3.11

If $f_1(w)$, $g_1(w)$ and $g_2(w)$ be any three bi-complex entire functions such that (i) $\rho_{f_1}(g_i) = \max\{\rho_{f_1}(g_k) \text{ for } k, i = 1,2\}$ and (ii) $\rho_{f_1}(g_1) \neq \rho_{f_1}(g_2)$ holds,

then $\sigma_{f_1}(g_1 . g_2) = \sigma_{f_1}(g_i), \text{for } i = 1,2.$

Lemma. 3.12

If $f_1(w)$, $g_1(w)$ and $g_2(w)$ be any three bi-complex entire functions such that (i) $\lambda_{f_1}(g_i) = \max\{\lambda_{f_1}(g_k) \text{ for } k, i = 1,2\}$ and (ii) $\lambda_{f_1}(g_1) \neq \lambda_{f_1}(g_2)$ holds, then $\tau_{f_1}(g_1 \pm g_2) = \tau_{f_1}(g_i)$, for $i = 1,2$.

Lemma. 3.13

If $f_1(w)$, $g_1(w)$ and $g_2(w)$ be any three bi-complex entire functions such that (i) $\lambda_{f_1}(g_i) = \max\{\lambda_{f_1}(g_k) \text{ for } k, i = 1,2\}$ and (ii) $\lambda_{f_1}(g_1) \neq \lambda_{f_1}(g_2)$ holds,

then $\tau_{f_1}(g_1 . g_2) = \tau_{f_1}(g_i)$, for $i = 1,2$.

4. THEOREMS

In this section we produce the main results of this paper.

Theorem 4.1.

If $f_1(w)$, $g_1(w)$ and $g_2(w)$ be any three bi-complex entire functions and if $\rho_{f_1}(g_i) = \max\{\rho_{f_1}(g_k) \text{ for } k, i = 1,2\}$, then $\rho_{f_1}(g_1 \pm g_2) \leq \rho_{f_1}(g_i), \text{for } i = 1,2$; where the equality holds when $\rho_{f_1}(g_1) \neq \rho_{f_1}(g_2)$.

Proof.

From the definition of relative order and relative lower order of entire functions, we have for all sufficiently large values of r that

$$M_{g_k}(r) \leq M_{f_k}\left(r^{\left(\rho_{f_k}(g_k)+\varepsilon\right)}\right) \tag{1}$$

$$and \quad M_{g_k}(r) \geq M_{f_k}\left(r^{\left(\lambda_{f_k}(g_k)-\varepsilon\right)}\right)$$

$$i.e. \quad M_{f_k}(r) \leq M_{g_k}\left(r^{\frac{1}{\left(\lambda_{f_k}(g_k)-\varepsilon\right)}}\right) \tag{2}$$

And also, for a sequence of values of r, tending to infinity, we get that

$$M_{g_k}(r) \geq M_{f_k}\left(r^{\left(\rho_{f_k}(g_k)-\varepsilon\right)}\right)$$

$$i.e. \quad M_{f_k}(r) \leq M_{g_k}\left(r^{\frac{1}{\left(\rho_{f_k}(g_k)-\varepsilon\right)}}\right) \tag{3}$$

$$and \;\; M_{g_k}(r) \leq \; M_{f_k}\left(r^{\left(\lambda_{f_k}(g_k)+\varepsilon\right)}\right), \tag{4}$$

$$where \; \varepsilon \; (> 0) \; isany \; arbitrary \; positive \; no. and \; i = 1, 2.$$

Case I. If $\rho_{f_1}(g_1 \pm g_2) = 0, then \;\; \rho_{f_1}(g_1 \pm g_2) \leq \; \rho_{f_1}(g_i), for \; i = 1{,}2$ is obvious.

So, let us suppose that $\rho_{f_1}(g_1 \pm g_2) > 0$. We can clearly assume that $\rho_{f_1}(g_i) \; for \; i = 1{,}2$ are finite. Also suppose that $\rho_{f_i}(g_1) < \; \rho_{f_k}(g_1), where \; k = i = 1{,}2 \; with \; f_i \; \neq f_k$ and g_1 is of regular relative growth w.r.t. at least one of f_1 or f_2 .

Now, in view of (1), (4) and Lemma 1, we obtain for a sequence of values of r tending to infinity that

$$M_{g_1 \pm g_2}(r) < \; M_{g_1}(r) + M_{g_2}(r)$$

$$i.e. \;\; M_{g_1 \pm g_2}(r) < \sum_{k=1}^{2} M_{f_1}(r^{\left(\rho_{f_k}(g_k)+\varepsilon\right)})$$

$$i.e. \;\; M_{g_1 \pm g_2}(r) < \; 2M_{f_1}(r^{\left(\rho_{f_k}(g_k)+\varepsilon\right)}$$

$$i.e. \;\; M_{g_1 \pm g_2}(r) < \; M_{f_1}(2r^{\left(\rho_{f_k}(g_k)+\varepsilon\right)}$$

$$i.e. \;\; log M_{f_1}^{-1} \, M_{g_1 \pm g_2}(r) < \; \left(\rho_{f_k}(g_k) + \varepsilon\right) \log \, r + O(1)$$

$$i.e. \;\; \frac{log M_{f_1}^{-1} \, M_{g_1 \pm g_2}(r)}{\log r} < \; \left(\rho_{f_k}(g_k) + \varepsilon\right) + \; \frac{O(1)}{\log r}$$

$$i.e. \;\; \lim_{r\to\infty} sup \; \frac{log M_{f_1}^{-1} \, M_{g_1 \pm g_2}(r)}{\log r} \leq \; \left(\rho_{f_k}(g_k) + \varepsilon\right)$$

Since, ε is arbitrary, it follows from the above that

$$\rho \;\; (g_1 \pm g_2) = \; \lim_{r\to\infty} sup \; \frac{log M_{f_1}^{-1} \, M_{g_1 \pm g_2}(r)}{\log r} \; \leq \; \rho_{f_k}(g_k) \, for \; k = 1, 2.$$

Now using the notion of the idempotent representation of bi-complex entire function, i.e. $f(w) = f_{e_1}.e_1 + f_{e_2}.e_2$, we can prove the above inequality separately for two different idempotent parts of the function f_1 in the same way, which proves the statement of the theorem.

Hence the proof.

Corollary 4.1.

If $f_1(w)$, $g_1(w)$ and $g_2(w)$ be any three bi-complex entire functions, then $\rho_{f_1}(g_1 \pm g_2) \leq \max\{\rho_{f_1}(g_i), for\ i = 1{,}2\}$ *; where the equality holds when* $\rho_{f_1}(g_1) \neq \rho_{f_1}(g_2)$.

Theorem 4.2.

If $f_1(w)$, $g_1(w)$ and $g_2(w)$ be any three bi-complex entire functions and if $\rho_{f_1}(g_i) = \max\{\rho_{f_1}(g_k)\ for\ k,i = 1{,}2\}$, *then* $\rho_{f_1}(g_1.g_2) \leq \rho_{f_1}(g_i), for\ i = 1{,}2$; *where the equality holds when* $\rho_{f_1}(g_1) \neq \rho_{f_1}(g_2)$.

Proof.

Case I. If $\rho_{f_1}(g_1.g_2) = 0, then\ \rho_{f_1}(g_1.g_2) \leq \rho_{f_1}(g_i), for\ i = 1{,}2$ is obvious.

So, let us suppose that $\rho_{f_1}(g_1.g_2) > 0$. We can clearly assume that $\rho_{f_1}(g_i)\ for\ i = 1{,}2$ are finite. Also suppose that $\rho_{f_i}(g_1) < \rho_{f_k}(g_1), where\ k = i = 1{,}2\ with\ f_i \neq f_k$ and g_1 is of regular relative growth w.r.t. at least one of f_1 *or* f_2 .

Now, in view of (1), (4) and Lemma 1, we obtain for a sequence of values of r tending to infinity and for any $\delta > 1$, we obtain that

$$\frac{1}{3} logM_{g_1.g_2}(r/2) \leq logM_{g_1}(r) + logM_{g_2}(r)$$

$$i.e.\ \frac{1}{3} logM_{g_1.g_2}(r/2) \leq \sum_{k=1}^{2} logM_{f_1}(r^{(\rho_{f_k}(g_k)+\varepsilon)})$$

$$i.e.\ \frac{1}{3} logM_{g_1.g_2}(r/2) \leq 2logM_{f_1}(r^{(\rho_{f_k}(g_k)+\varepsilon)})$$

$$i.e.\ \log M_{g_1.g_2}(r/2) \leq 6logM_{f_1}(r^{(\rho_{f_k}(g_k)+\varepsilon)})$$

$$i.e.\ M_{g_1.g_2}(r/2) \leq M_{f_1}[(r^{(\rho_{f_k}(g_k)+\varepsilon)}]^6$$

$$i.e.\ M_{g_1.g_2}(r/2) \leq M_{f_1} r^{\delta(\rho_{f_k}(g_k)+\varepsilon)}$$

$$i.e.\ logM_{f_1}^{-1} M_{g_1.g_2}(r/2) < \delta(\rho_{f_k}(g_k) + \varepsilon) \log r$$

$$i.e.\ \frac{logM_{f_1}^{-1} M_{g_1.g_2}(r/2)}{\log(\frac{r}{2})} < \frac{\delta(\rho_{f_k}(g_k) + \varepsilon) \log r}{\log r + O(1)}$$

Since, $\varepsilon > 0$ is arbitrary, we get from the above by letting $\delta \to 1+$, that

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$$\rho_{f_1}(g_1.g_2) = \lim_{r\to\infty} sup \frac{log M_{f_1}^{-1} M_{g_1.g_2}(r)}{\log r} \leq \rho_{f_k}(g_k)\, for\ k = 1, 2.$$

Now using the notion of the idempotent representation of bi-complex entire function, i.e. $f(w) = f_{e_1}.e_1 + f_{e_2}.e_2$, we can prove the above inequality separately for two different idempotent parts of the function f_1 in the same way, which proves the statement of the theorem.

Hence the proof.

Corollary 4.2. *If $f_1(w)$, $g_1(w)$ and $g_2(w)$ be any three bi-complex entire functions and if $\rho_{f_1}(g_i) = \max\{\rho_{f_1}(g_k)\, for\ k, i = 1,2\}$, then $\rho_{f_1}(g_1.g_2) \leq \max\{\rho_{f_1}(g_i), for\ i = 1,2$; where the equality holds when $\rho_{f_1}(g_1) \neq \rho_{f_1}(g_2)$.*

Theorem 4.3.

If $f_1(w)$, $g_1(w)$ and $g_2(w)$ be any three bi-complex entire functions such that at least one of g_1 and g_2 is of regular relative growth with respect to f_1 and $\lambda_{f_1}(g_i) = \max\{\lambda_{f_1}(g_k)\, for\ k, i = 1,2\}$, then $\lambda_{f_1}(g_1 \pm g_2) \leq \lambda_{f_1}(g_i), for\ i = 1,2$; where the equality holds when $\lambda_{f_1}(g_1) \neq \lambda_{f_1}(g_2)$.

Proof.

From the definition of relative order and relative lower order of entire functions, we have for all sufficiently large values of r that

$$M_{g_k}(r) \leq M_{f_k}\left(r^{\left(\rho_{f_k}(g_k)+\varepsilon\right)}\right) \tag{5}$$

$$nd\ \ M_{g_k}(r) \geq M_{f_k}\left(r^{\left(\lambda_{f_k}(g_k)-\varepsilon\right)}\right)$$

$$i.e.\ \ M_{f_k}(r) \leq M_{g_k}\left(r^{\frac{1}{\left(\lambda_{f_k}(g_k)-\varepsilon\right)}}\right) \tag{6}$$

And also, for a sequence of values of r, tending to infinity, we get that

$$M_{g_k}(r) \geq M_{f_k}\left(r^{\left(\rho_{f_k}(g_k)-\varepsilon\right)}\right)$$

$$i.e.\ \ M_{f_k}(r) \leq M_{g_k}\left(r^{\frac{1}{\left(\rho_{f_k}(g_k)-\varepsilon\right)}}\right) \tag{7}$$

$$and \;\; M_{g_k}(r) \le M_{f_k}\left(r^{\left(\lambda_{f_k}(g_k)+\varepsilon\right)}\right), \tag{8}$$

$$where\ \varepsilon\ (>0)\ is any\ arbitrary\ positive\ no.\ and\ i = 1, 2.$$

Case I. If $\rho_{f_1}(g_1 \pm g_2) = 0, then\ \rho_{f_1}(g_1 \pm g_2) \le \rho_{f_1}(g_i), for\ i = 1,2$ is obvious.

So, let us suppose that $\rho_{f_1}(g_1 \pm g_2) > 0$. We can clearly assume that $\rho_{f_1}(g_i)\ for\ i = 1,2$ are finite. Also suppose that $\rho_{f_i}(g_1) < \rho_{f_k}(g_1), where\ k = i = 1,2\ with\ f_i \neq f_k$ and g_1 is of regular relative growth w.r.t. at least one of f_1 or f_2 .

Now, in view of (5), (8) and Lemma 1, we obtain for a sequence of values of r tending to infinity that

$$M_{g_1 \pm g_2}(r) < M_{g_1}(r) + M_{g_2}(r)$$

$$i.e.\quad M_{g_1 \pm g_2}(r) < \sum_{i=1}^{2} M_{f_1}(r^{(\lambda_{f_1}(g_i)+\varepsilon)})$$

$$i.e.\quad M_{g_1 \pm g_2}(r) < 2M_{f_1}(r^{\left(\lambda_{f_i}(g_i)+\varepsilon\right)}$$

$$i.e.\quad M_{g_1 \pm g_2}(r) < M_{f_1}(2r^{\left(\lambda_{f_i}(g_i)+\varepsilon\right)}$$

$$i.e.\ \ log M_{f_1}^{-1} M_{g_1 \pm g_2}(r) < \left(\lambda_{f_1}(g_i) + \varepsilon\right) \log r + O(1)$$

$$i.e.\quad \frac{log M_{f_1}^{-1} M_{g_1 \pm g_2}(r)}{\log r} < \left(\lambda_{f_1}(g_i) + \varepsilon\right) + \frac{O(1)}{\log r}$$

$$i.e.\ \ \liminf_{r\to\infty} \frac{log M_{f_1}^{-1} M_{g_1 \pm g_2}(r)}{\log r} \le \left(\lambda_{f_1}(g_i) + \varepsilon\right)$$

Since, ε is arbitrary, it follows from the above that

$$\lambda_{f_1}(g_1 \pm g_2) = \liminf_{r\to\infty} \frac{log M_{f_1}^{-1} M_{g_1 \pm g_2}(r)}{\log r} \le \lambda_{f_1}(g_k)\ for\ k = 1, 2.$$

Now using the notion of the idempotent representation of bi-complex entire function, i.e. $f(w) = f_{e_1}.e_1 + f_{e_2}.e_2$, we can prove the above inequality separately for two different idempotent parts of the function f_1 in the same way, which proves the statement of the theorem.

Hence the proof.

Corollary 4.3.

If $f_1(w)$, $g_1(w)$ and $g_2(w)$ be any three bi-complex entire functions, then $\lambda_{f_1}(g_1 \pm g_2) \leq \min\{\lambda_{f_1}(g_i), for\ i = 1,2\}$; *where the equality holds when* $\lambda_{f_1}(g_1) \neq \lambda_{f_1}(g_2)$.

Theorem 4.4.

If $f_1(w)$, $g_1(w)$ and $g_2(w)$ be any three bi-complex entire functions such that f_1 has the Property(A) and at least one of g_1 and g_2 is of regular relative growth with respect to f_1 and $\lambda_{f_1}(g_i) = \max\{\lambda_{f_1}(g_k)\ for\ k, i = 1,2\}$, *then* $\lambda_{f_1}(g_1.g_2) \leq \lambda_{f_1}(g_i), for\ i = 1,2$; *where the equality holds when* $\lambda_{f_1}(g_1) \neq \lambda_{f_1}(g_2)$.

Proof.

Case I. If $\lambda_{f_1}(g_1.g_2) = 0, then\ \lambda_{f_1}(g_1.g_2) \leq \lambda_{f_1}(g_i), for\ i = 1,2$ is obvious.

So, let us suppose that $\lambda_{f_1}(g_1.g_2) > 0$. We can clearly assume that $\lambda_{f_1}(g_i)\ for\ i = 1,2$ are finite. Also suppose that $\lambda_{f_i}(g_1) < \lambda_{f_k}(g_1), where\ k = i = 1,2\ with\ f_i \neq f_k$ and g_1 is of regular relative growth w.r.t. at least one of f_1 *or* f_2 .

Now, in view of (1), (4) and Lemma 2, we obtain for a sequence of values of r tending to infinity and for any $\delta > 1$, we obtain that

$$\frac{1}{3} logM_{g_1.g_2}(r/2) \leq logM_{g_1}(r) + logM_{g_2}(r)$$

$$i.e.\ \frac{1}{3} logM_{g_1.g_2}(r/2) \leq \sum_{k=1}^{2} logM_{f_1}(r^{(\lambda_{f_1}(g_k)+\varepsilon)})$$

$$i.e.\ \frac{1}{3} logM_{g_1.g_2}(r/2) \leq 2logM_{f_1}(r^{(\lambda_{f_1}(g_k)+\varepsilon)})$$

$$i.e.\ \log M_{g_1.g_2}(r/2) \leq 6logM_{f_1}(r^{(\lambda_{f_1}(g_k)+\varepsilon)})$$

$$i.e.\ M_{g_1.g_2}(r/2) \leq M_{f_1}[(r^{(\lambda_{f_1}(g_k)+\varepsilon)}]^6$$

$$i.e.\ M_{g_1.g_2}(r/2) \leq M_{f_1} r^{\delta(\lambda_{f_1}(g_k)+\varepsilon)}$$

$$i.e.\ logM_{f_1}^{-1} M_{g_1.g_2}(r/2) < \delta(\lambda_{f_1}(g_k) + \varepsilon) \log r$$

$$i.e.\ \frac{logM_{f_1}^{-1} M_{g_1.g_2}(r/2)}{\log(\frac{r}{2})} < \frac{\delta(\lambda_{f_1}(g_k) + \varepsilon) \log r}{\log r + O(1)}$$

Since, $\varepsilon > 0$ is arbitrary, we get from the above by letting $\delta \to 1+$, that

$$\lambda_{f_1}(g_1.g_2) = \liminf_{r\to\infty} \frac{\log M_{f_1}^{-1} M_{g_1.g_2}(r)}{\log r} \leq \lambda_{f_1}(g_k)\ for\ k = 1,2.$$

Now using the notion of the idempotent representation of bi-complex entire function, i.e. $f(w) = f_{e_1}.e_1 + f_{e_2}.e_2$, we can prove the above inequality separately for two different idempotent parts of the function in the same way, which proves the statement of the theorem.

Hence the proof.

Corollary 4.4. *If $f_1(w)$, $g_1(w)$ and $g_2(w)$ be any three bi-complex entire functions such that f_1 has the Property(A) and at least one of g_1 and g_2 is of regular relative growth with respect to f_1 and* $\lambda_{f_1}(g_i) = \max\{\lambda_{f_1}(g_k)\ for\ k,i = 1,2\}$, *then* $\lambda_{f_1}(g_1.g_2) \leq \min\{\lambda_{f_1}(g_i), for\ i = 1,2\}$; *where the equality holds when* $\lambda_{f_1}(g_1) \neq \lambda_{f_1}(g_2)$.

Theorem 4.5.

If $f_1(w)$, $g_1(w)$ and $g_2(w)$ be any three bi-complex entire functions such that $(i)\ \rho_{f_1}(g_i) = \max\{\rho_{f_1}(g_k)\ for\ k,i = 1,2\}$ *and* $(ii)\ \rho_{f_1}(g_1) \neq \rho_{f_1}(g_2)$ *holds,*

then $\sigma_{f_1}(g_1 \pm g_2) = \sigma_{f_1}(g_i), for\ i = 1,2.$

Proof. From the definition of relative type and relative lower type of entire function, we have for all sufficiently large values of r that

$$M_{g_k}(r) \leq M_{f_k}\left[(\sigma_{f_k}(g_k) + \epsilon)\, r^{\rho_{f_k}(g_k)}\right] \tag{9}$$

$$and\ \ M_{g_k}(r) \geq M_{f_k}\left[(\bar{\sigma}_{f_k}(g_k) - \epsilon)\, r^{\rho_{f_k}(g_k)}\right]$$

$$i.e.\ \ M_{f_k}(r) \geq M_{g_k}\left[\left(\frac{r}{\bar{\sigma}_{f_k}(g_k)-\epsilon}\right)^{\frac{1}{\rho_{f_k}(g_k)}}\right], \tag{10}$$

and also for a sequence $\{r_n\}$ of values of r tending to infinity, we get

$$M_{g_k}(r) \geq M_{f_k}\left[\sigma_{f_k}(g_k) - \epsilon)\, r_n^{\rho_{f_k}(g_k)}\right]$$

$$i.e.\ \ M_{f_k}(r) \leq M_{g_k}\left[\left(\frac{r}{\sigma_{f_k}(g_k)-\epsilon}\right)^{\frac{1}{\rho_{f_k}(g_k)}}\right] \tag{11}$$

$$M_{g_k}(r) \leq M_{f_k}\left[(\bar{\sigma}_{f_k}(g_k)+\epsilon)\, r_n{}^{\rho_{f_k}(g_k)}\right], \qquad (12)$$

where $\epsilon > 0$ is any arbitrary small positive number and k=1, 2.

Let $\rho_{f_1}(g_i) < \rho_{f_1}(g_k)$, where $k, i = 1,2$ with $g_k \neq g_i$ for $k \neq i$.

Now from (9) and (12) we get for a sequence $\{r_n\}$ of values of r tending to infinity that

$$M_{g_1 \pm g_2}(r_n) < M_{g_1}(r_n) + M_{g_2}(r_n) \text{ which implies that}$$

$$M_{g_1 \pm g_2}(r_n) < M_{f_1}\left[(\sigma_{f_1}(g_k)+\epsilon)\, r_n{}^{\rho_{f_1}(g_k)}\right] + M_{f_1}\left[(\bar{\sigma}_{f_1}(g_i)+\epsilon)\, r_n{}^{\rho_{f_1}(g_i)}\right]$$

Therefore $$M_{g_1 \pm g_2}(r_n) < M_{f_1}\left[(\sigma_{f_1}(g_k)+\epsilon)\, r_n{}^{\rho_{f_1}(g_k)}\right]\left[1 + \frac{M_{f_1}\left[(\bar{\sigma}_{f_1}(g_i)+\epsilon)\, r_n{}^{\rho_{f_1}(g_i)}\right]}{M_{f_1}\left[(\sigma_{f_1}(g_k)+\epsilon)\, r_n{}^{\rho_{f_1}(g_k)}\right]}\right]$$

Since $\rho_{f_1}(g_i) < \rho_{f_1}(g_k)$, the term $\frac{M_{f_1}\left[(\bar{\sigma}_{f_1}(g_i)+\epsilon)\, r_n{}^{\rho_{f_1}(g_i)}\right]}{M_{f_1}\left[(\sigma_{f_1}(g_k)+\epsilon)\, r_n{}^{\rho_{f_1}(g_k)}\right]} \to 0$ when n is sufficiently large.

Therefore, in view of Lemma (1) and the above, we get for a sequence of values $\{r_n\}$ of r tending to infinity that

$$M_{g_1 \pm g_2}(r_n) < M_{f_1}\left[(\sigma_{f_1}(g_k)+\epsilon)\, r_n{}^{\rho_{f_1}(g_k)}\right](1+\varepsilon_1)$$

$$i.e.\ M_{g_1 \pm g_2}(r_n) < M_{f_1}\left[\alpha(\sigma_{f_1}(g_k)+\epsilon)\, r_n{}^{\rho_{f_1}(g_k)}\right], \text{ where } \alpha > 1+\epsilon_1.$$

Now, making $\alpha \to 1$, we obtain for a sequence $\{r_n\}$ of values of r tending to infinity that

$$i.e.\ M_{f_1}^{-1} M_{g_1 \pm g_2}(r_n) < (\sigma_{f_1}(g_k)+\epsilon)\, r_n{}^{\rho_{f_1}(g_k)}$$

$$i.e.\ \frac{M_{f_1}^{-1} M_{g_1 \pm g_2}(r_n)}{r_n{}^{\rho_{f_1}(g_1 \pm g_2)}} < \sigma_{f_1}(g_k)+\epsilon$$

Since $\epsilon > 0$ is arbitrary, we get $\sigma_{f_1}(g_1 \pm g_2) \leq \sigma_{f_1}(g_k), for\ k = 1,2.$

Further without loss of generality, let $\rho_{f_1}(g_1) < \rho_{f_1}(g_2)$ and $g = g_1 \pm g_2$.

Then $\sigma_{f_1}(g) \leq \sigma_{f_1}(g_2)$ and $\rho_{f_1}(g_1) < \rho_{f_1}(g)$. Therefore, $\sigma_{f_1}(g_2) \leq \sigma_{f_1}(g)$.

Hence $\sigma_{f_1}(g) \leq \sigma_{f_1}(g_2)$ *which implies* $\sigma_{f_1}(g_1 \pm g_2) = \sigma_{f_1}(g_2)$.

Thus $\sigma_{f_1}(g_1 \pm g_2) = \sigma_{f_1}(g_i)\ for\ i = 1,2.$

Now using the notion of the idempotent representation of bi-complex entire function, i.e. $f(w) = f_{e_1}.e_1 + f_{e_2}.e_2$, we can prove the above inequality separately for two different idempotent parts of the function f_1 in the same way, which proves the statement of the theorem.

Hence the proof.

Theorem 4.6.

If $f_1(w)$, $g_1(w)$ and $g_2(w)$ be any three bi-complex entire functions such that $(i)\ \rho_{f_1}(g_i) = max\ \{\rho_{f_1}(g_k)\ for\ \ k,i = 1,2\}$ and $(ii)\ \rho_{f_1}(g_1) \neq \rho_{f_1}(g_2)$ holds ,

then $\sigma_{f_1}(g_1.g_2) = \sigma_{f_1}(g_i), for\ \ i = 1,2.$

Proof. Let us take $\rho_{f_1}(g_k) < \rho_{f_1}(g_i)$ for i, k = 1, 2 with $g_k \neq g_i$.

Now for any arbitrary $\epsilon > 0$, we have for all sufficiently large values of r that

$$M_{g_1.g_2}(r) \leq M_{g_1}(r).M_{g_2}(r)$$

Therefore, $M_{g_1.g_2}(r) \leq M_{f_1}\left[(\sigma_{f_1}(g_k) + \epsilon/2)\, r^{\rho_{f_1}(g_k)}\right].M_{f_1}\left[(\sigma_{f_1}(g_i) + \right.$ □/2) □□□1□□

Since $\rho_{f_1}(g_k) < \rho_{f_1}(g_i)\ for\ \ i,k = 1,2$, we get for all sufficiently large values of r that

$$(\sigma_{f_1}(g_i) + \epsilon)r^{\rho_{f_1}(g_i)} > (\sigma_{f_1}(g_k) + \epsilon)\, r^{\rho_{f_1}(g_k)}.$$

Therefore, $M_{f_1}(\sigma_{f_1}(g_i) + \epsilon)r^{\rho_{f_1}(g_i)} > M_{f_1}(\sigma_{f_1}(g_k) + \epsilon)\, r^{\rho_{f_1}(g_k)}.$

Therefore, from the above arguments we get

$$M_{g_1.g_2}(r) \leq M_{f_1}\left[(\sigma_{f_1}(g_i) + \epsilon/2)r^{\rho_{f_1}(g_i)}\right]^2 \tag{13}$$

We have $\frac{\sigma_{f_1}(g_i)+\epsilon}{\sigma_{f_1}(g_i)+\epsilon/2} = \delta_1\ (say\ \ ) > 1 which\ implies$

$$log\ \ (\sigma_{f_1}(g_i) + \epsilon)r^{\rho_{f_1}(g_i)} > log\ \ (\sigma_{f_1}(g_i) + \epsilon/2)\, r^{\rho_{f_1}(g_i)}$$

$$i.e.\ \frac{log\ \ (\sigma_{f_1}(g_i) + \epsilon)r^{\rho_{f_1}(g_i)}}{log\ \ (\sigma_{f_1}(g_i) + \epsilon/2)\, r^{\rho_{f_1}(g_i)}} = \delta\ (say\ \ ) > 1$$

$$i.e.\ log\ \ (\sigma_{f_1}(g_i) + \epsilon)r^{\rho_{f_1}(g_i)} = \delta.log\ \ (\sigma_{f_1}(g_i) + \epsilon/2)\, r^{\rho_{f_1}(g_i)} \tag{14}$$

Since f_1 has the Property (A), in view of Lemma 2, we obtain from (13) for all sufficiently large values of r that

$$M_{g_1.g_2}(r) < M_{f_1}\left[(\sigma_{f_1}(g_i) + \epsilon/2)r^{\rho_{f_1}(g_i)}\right]^{\delta}$$

$$\text{i.e. } M_{g_1.g_2}(r) < M_{f_1}\left[(\sigma_{f_1}(g_i)+\epsilon)r^{\rho_{f_1}(g_i)}\right]$$

$$\text{i.e. } \frac{M_{f_1}{}^{-1} M_{g_1.g_2}(r)}{r^{\rho_{f_1}(g_i)}} < [(\sigma_{f_1}(g_i)+\epsilon)]$$

$$\text{i.e. } \frac{M_{f_1}{}^{-1} M_{g_1.g_2}(r)}{r^{\rho_{f_1}(g_1.g_2)}} < [(\sigma_{f_1}(g_i)+\epsilon)]$$

$$\sigma_{f_1}(g_1.g_2) \leq \sigma_{f_1}(g_i), for \quad i = 1,2. \tag{19}$$

Further without loss of generality, let $g = g_1.g_2$ and $\rho_{f_1}(g_1) < \rho_{f_1}(g_2) = \rho_{f_1}(g)$.

Then $\sigma_{f_1}(g) \leq \sigma_{f_1}(g_2)$ which implies $\sigma_{f_1}(g_2) \leq \sigma_{f_1}(g)$ as $g_2 = \frac{g}{g_1}$.

Hence $\sigma_{f_1}(g) = \sigma_{f_1}(g_2)$ which implies $\sigma_{f_1}(g_1.g_2) \leq \sigma_{f_1}(g_2)$.

Thus $\sigma_{f_1}(g_1.g_2) = \sigma_{f_1}(g_i), for \quad i = 1,2.$

Now using the notion of the idempotent representation of bi-complex entire function, i.e. $f(w) = f_{e_1}.e_1 + f_{e_2}.e_2$, we can prove the above inequality separately for two different idempotent parts of the function f_1 in the same way, which proves the statement of the theorem.

Hence the proof.

Theorem 4.7.

If $f_1(w)$, $g_1(w)$ and $g_2(w)$ be any three bi-complex entire functions such that (i) $\lambda_{f_1}(g_i) = \max\{\lambda_{f_1}(g_k) \text{ for } k, i = 1,2\}$ and (ii) $\lambda_{f_1}(g_1) \neq \lambda_{f_1}(g_2)$ holds, then $\tau_{f_1}(g_1 \pm g_2) = \tau_{f_1}(g_i)$, for $i = 1,2$.

Proof. From the definition of relative type and relative lower type of entire function, we have for all sufficiently large values of r that

$$M_{g_k}(r) \leq M_{f_k}\left[(\overline{\tau}_{f_k}(g_k)+\epsilon)\, r^{\lambda_{f_k}(g_k)}\right] \tag{20}$$

$$and \quad M_{g_k}(r) \geq M_{f_k}\left[(\tau_{f_k}(g_k)-\epsilon)\, r^{\lambda_{f_k}(g_k)}\right]$$

$$i.e.\ M_{f_k}(r) \geq M_{g_k}\left[\left(\frac{r}{\tau_{f_k}(g_k)-\epsilon}\right)^{\frac{1}{\lambda_{f_k}(g_k)}}\right], \tag{21}$$

and also for a sequence $\{r_n\}$ of values of r tending to infinity, we get

$$M_{g_k}(r) \geq M_{f_k}\left[\overline{\tau}_{f_k}(g_k) - \epsilon) r_n^{\lambda_{f_k}(g_k)}\right]$$

$$i.e.\ M_{f_k}(r) \leq M_{g_k}\left[\left(\frac{r}{\overline{\tau}_{f_k}(g_k)-\epsilon}\right)^{\frac{1}{\lambda_{f_k}(g_k)}}\right] \tag{22}$$

$$M_{g_k}(r) \leq M_{f_k}\left[(\tau_{f_k}(g_k) + \epsilon) r_n^{\lambda_{f_k}(g_k)}\right], \tag{23}$$

where $\epsilon > 0$ is any arbitrary small positive number and k=1, 2.

Let $\lambda_{f_1}(g_k) < \lambda_{f_1}(g_i)$, where $k, i = 1,2$ with $g_k \neq g_i$ for $k \neq i$.

Now from (20) and (21) we get for a sequence $\{r_n\}$ of values of r tending to infinity that

$$M_{g_1 \pm g_2}(r_n) < M_{g_1}(r_n) + M_{g_2}(r_n) \text{ which implies that}$$

$$M_{g_1 \pm g_2}(r_n) < M_{f_1}\left[(\overline{\tau}_{f_1}(g_k) + \epsilon) r_n^{\lambda_{f_1}(g_k)}\right] + M_{f_1}\left[(\tau_{f_1}(g_i) + \epsilon) r_n^{\lambda_{f_1}(g_i)}\right]$$

Therefore

$$M_{g_1 \pm g_2}(r_n) < M_{f_1}\left[(\tau_{f_1}(g_k) + \epsilon) r_n^{\lambda_{f_1}(g_k)}\right]\left[1 + \frac{M_{f_1}\left[(\overline{\tau}_{f_1}(g_i)+\epsilon) r_n^{\lambda_{f_1}(g_i)}\right]}{M_{f_1}\left[(\tau_{f_1}(g_k)+\epsilon) r_n^{\lambda_{f_1}(g_k)}\right]}\right]$$

Since $\lambda_{f_1}(g_k) < \lambda_{f_1}(g_i)$, the term $\frac{M_{f_1}\left[(\overline{\tau}_{f_1}(g_i)+\epsilon) r_n^{\lambda_{f_1}(g_i)}\right]}{M_{f_1}\left[(\tau_{f_1}(g_k)+\epsilon) r_n^{\lambda_{f_1}(g_k)}\right]} \to 0$ when n is sufficiently large.

Therefore, in view of Lemma (1) and the above, and using the same technique as in Theorem 5, we get for a sequence of values $\{r_n\}$ of r tending to infinity that

$$\frac{M_{f_1}^{-1} M_{g_1 \pm g_2}(r_n)}{r_n^{\lambda_{f_1}(g_1 \pm g_2)}} < \tau_{f_1}(g_k) + \epsilon$$

Since $\epsilon > 0$ is arbitrary, we get $\tau_{f_1}(g_1 \pm g_2) \leq \tau_{f_1}(g_k), for \quad k = 1,2.$

Further without loss of generality, let $\lambda_{f_1}(g_1) < \lambda_{f_1}(g_2)$ and $g = g_1 \pm g_2$.

Then $\tau_{f_1}(g) \leq \tau_{f_1}(g_2)$ and $\tau_{f_1}(g_1) < \tau_{f_1}(g)$. Therefore, $\tau_{f_1}(g_2) \leq \tau_{f_1}(g)$.

Hence $\tau_{f_1}(g) \leq \tau_{f_1}(g_2)$ *which implies* $\tau_{f_1}(g_1 \pm g_2) = \tau_{f_1}(g_2)$.

Thus $\tau_{f_1}(g_1 \pm g_2) = \tau_{f_1}(g_i)\ for \quad i = 1,2.$

Now using the notion of the idempotent representation of bi-complex entire function, i.e. $f(w) = f_{e_1}.e_1 + f_{e_2}.e_2$, we can prove the above inequality separately for two different idempotent parts of the function f_1 in the same way, which proves the statement of the theorem.

Hence the proof.

Theorem 4.8.

If $f_1(w)$, $g_1(w)$ and $g_2(w)$ be any three bi-complex entire functions such that (i) $\lambda_{f_1}(g_i) = \max\{\lambda_{f_1}(g_k)$ for $k, i = 1,2\}$ and (ii) $\lambda_{f_1}(g_1) \neq \lambda_{f_1}(g_2)$ holds,

then $\tau_{f_1}(g_1.g_2) = \tau_{f_1}(g_i)$, for $i = 1,2$.

Proof.

Let us take $\lambda_{f_1}(g_k) < \lambda_{f_1}(g_i)$ for $i, k = 1, 2$ with $g_k \neq g_i$ with atleast one of each has a regular relative growth w.r.t. f_1 .

Now for any arbitrary $\epsilon > 0$, we obtain for a sequence of values $\{r_n\}$ of r tending to infinity that

$$M_{g_1.g_2}(r_n) \leq M_{g_1}(r_n) . M_{g_2}(r_n)$$

Therefore, $M_{g_1.g_2}(r_n) \leq M_{f_1}\left[(\overline{\tau}_{f_1}(g_k) + \epsilon/2)\, r^{\lambda_{f_1}(g_k)}\right] . M_{f_1}[(\tau_{f_1}(g_i) +$ □/2) □□□1□□

Since $\lambda_{f_1}(g_k) < \lambda_{f_1}(g_i)$ *for* $i, k = 1,2$, we get for a sequence of values $\{r_n\}$ of r tending to infinity that

$$M_{g_1.g_2}(r_n) \leq M_{f_1}\left[(\tau_{f_1}(g_i) + \epsilon/2) r^{\lambda_{f_1}(g_i)}\right]^2 \tag{24}$$

Now using the similar technique as followed in the proof of Theorem 6, we have from the above

$$\frac{M_{f_1}^{-1} M_{g_1.g_2}(r_n)}{r^{\lambda_{f_1}(g_1.g_2)}} < [(\tau_{f_1}(g_i) + \epsilon)]$$

Therefore, $\tau_{f_1}(g_1.g_2) \leq \tau_{f_1}(g_i)$, *for* $i = 1,2.$

(25)

Further without loss of generality, let $g = g_1.g_2$ and $\lambda_{f_1}(g_1) < \lambda_{f_1}(g_2) = \lambda_{f_1}(g)$.

Then $\tau_{f_1}(g) \leq \tau_{f_1}(g_2)$ which implies $\tau_{f_1}(g_2) \leq \tau_{f_1}(g)$ *as* $g_2 = \frac{g}{g_1}$.

Hence $\tau_{f_1}(g) = \tau_{f_1}(g_2)$ which implies $\tau_{f_1}(g_1.g_2) \leq \tau_{f_1}(g_2)$.

Thus $\tau_{f_1}(g_1.g_2) = \tau_{f_1}(g_i), for \quad i = 1,2.$

Now using the notion of the idempotent representation of bi-complex entire function, i.e. $f(w) = f_{e_1}.e_1 + f_{e_2}.e_2$, we can prove the above inequality separately for two different idempotent parts of the function f_1 in the same way, which proves the statement of the theorem.

Hence the proof.

5. CONCLUSIONS

In this paper, we have investigated certain growth properties regarding relative order (lower order), relative type (lower type) and relative weak type of entire functions. Basically, we have worked on the sum and product theorems of two or more entire functions. However, the treatment of these notions may also be extended for bi-complex meromorphic functions, in the field of slowly changing functions also in case of entire or meromorphic functions of several variables. Interested mathematicians and researchers in this field may be interested to go through several books and research papers already published around the world for their further study.

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