International Journal of Engineering, Science and Mathematics
Vol. 9 Issue 7, July 2020,
ISSN: 2320-0294 Impact Factor: 6.765
Journal Homepage: http://www.ijesm.co.in, Email: ijesmj@gmail.com
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PRODUCT OF SOME SPECIAL FUNCTIONS ASSOCIATE BY MARICHEV – SAIGO –MAEDA FRACTIONAL INTEGRAL OPERATORS

Hemlata Saxena

KEYWORDS:

Marichev-Saigo-Maeda fractional integral operators, $\overline{H}$ - function, Generalized Wright hypergeometric function,Generalised Mittag-Leffler function, Generalized polynomial set.

ABSTRACT

In this paper we introduce generalized fractional calculus operators which involves the Appell's function $F_3(\,.\,)$ to the product of the $\overline{H}$ -function, generalized Mittag Leffler function and generalized polynomial set. In this we have considered some special cases of our main results. Some known fractional integrals and derivatives are also included in it.

Author correspondence:

Hemlata Saxena,
Department of Mathematics, Career Point University,
Kota-324005, Rajasthan, India.

1. INTRODUCTION

Fractional integral operators are widely used to solve differential equations and integral equations. So a lot of work has been done on the theory and applications of Fractional integral operators. Most popular fractional integral transforms are due to Kalla and Saxena [4], Kiryakova [5, 6] etc. have studied in depth. In this paper using fractional integral operator gives new challenges for interpretations of its operations not only in physical and theoretical models but also the model of real phenomena and events in mathematics. A useful generalized of the hypergeometric fractional integrals has been extended by Saigo and Maeda [8,Pg.386-400 eq. (4.12) and (4.13)] in terms of any complex order with Appell's function $F_3(\,.\,)$ in the Kernel, as follows :

Let $\alpha,\alpha',\beta,\beta'$, $\gamma\in C$ and $x>0$ then the generalized fractional calculus operators (the Marichev-Saigo-Maeda operators) involving the Appell's function[9] is defined as :

$$\left(I_{o,+}^{\alpha,\alpha',\beta,\beta',\gamma} f\right)(x)=\frac{x^{-\alpha}}{\Gamma(\gamma)}\int_0^x (x-t)^{\gamma-1} t^{-\alpha'} F_3\left(\alpha,\alpha',\beta,\beta';\gamma;1-\frac{t}{x},1-\frac{x}{t}\right) f(t)\,dt$$

$$\left(R(\gamma)>0\right) \qquad .....(1)$$

and

$$\left(I_{o,-}^{\alpha,\alpha',\beta,\beta',\gamma} f\right)(x)=\frac{x^{-\alpha'}}{\Gamma(\gamma)}\int_x^\infty (t-x)^{\gamma-1} t^{-\alpha} F_3\left(\alpha,\alpha',\beta,\beta';\gamma;1-\frac{x}{t},1-\frac{t}{x}\right) f(t)\,dt$$

$$\left(R(\gamma)>0\right) \qquad .....(2)$$

The left hand side and right hand side generalized integration of the type (1) and (2) for a power function are giving by :

$$\left(I_{0,+}^{\alpha,\alpha',\beta,\beta',\gamma} x^{\rho-1}\right)(x)=\Gamma\left[\begin{matrix}\rho,\rho+\gamma-\alpha-\alpha'-\beta,\rho+\beta'-\alpha'\\ \rho+\beta',\rho+\gamma-\alpha-\alpha',\rho+\gamma-\alpha'-\beta\end{matrix}\right]x^{\rho-\alpha-\alpha'+\gamma-1}$$

$$.....(3)$$

where $\mathrm{Re}(\gamma)>0$, $\mathrm{Re}(\rho)>\max\left\{0,\mathrm{Re}(\alpha+\alpha'+\beta-\gamma),\mathrm{Re}(\alpha'-\beta')\right\}$ and

$$\left(I_{0,-}^{\alpha,\alpha',\beta,\beta',\gamma} x^{\rho-1}\right)(x)=\Gamma\left[\begin{matrix}1-\rho-\gamma+\alpha+\alpha',1-\rho+\alpha+\beta'-\gamma,1-\rho-\beta\\ 1-\rho,1-\rho+\alpha+\alpha'+\beta'-\gamma,1-\rho+\alpha-\beta\end{matrix}\right]x^{\rho-\alpha-\alpha'+\gamma-1}$$

$$.....(4)$$

where $\operatorname{Re}(\gamma)>0,\ \operatorname{Re}(\rho)<1+\min\{\operatorname{Re}(-\beta),\operatorname{Re}(\alpha+\alpha'-\gamma),\operatorname{Re}(\alpha+\beta'-\gamma)\}$ the symbol occurring in (3) and (4) is given by

$$\Gamma\begin{bmatrix} a,b,c \\ d,e,f \end{bmatrix}=\frac{\Gamma(a)\,\Gamma(b)\Gamma(c)}{\Gamma(d)\,\Gamma(e)\,\Gamma(f)} \qquad .....(5)$$

Generalized Mittag-Leffler function defined by Prabhakar [11] is follows :

$$E_{\alpha,\beta}^{\gamma}(z)=\sum_{n=0}^{\infty}\frac{(\gamma)_n\,.z^n}{\Gamma(\alpha n+\beta)n!} \qquad .....(6)$$

where $(\alpha,\beta,\gamma,\in C,\operatorname{Re}(\alpha)>0,\ \operatorname{Re}(\beta)>0,\operatorname{Re}(\gamma)>0)$

—

H – function

The $\overline{H}$-function, defined as follows which is introduced by Inayat-Hussain [1, 2] and studied by Buschman and Srivastava [7] and others.

$$\overline{H}_{P,Q}^{M,N}[Z]=\overline{H}_{P,Q}^{M,N}\left[Z\left|\begin{matrix}(e_j,E_j;a_j)_{1,N},(e_j,E_j)_{N+1,P}\\(f_j,F_j)_{1,M},(f_j,F_j;b_j)_{M+1,Q}\end{matrix}\right.\right]=\frac{1}{2\pi i}\int_L\theta(s)z^S ds\,, \qquad .....(7)$$

where

$$\theta(s)=\frac{\prod_{j=1}^{M}\Gamma(f_j-F_js)\prod_{j=1}^{N}\{\Gamma(1-e_j+E_js)\}^{a_j}}{\prod_{j=M+1}^{Q}\{\Gamma(1-f_j+F_js)\}^{b_j}\prod_{j=N+1}^{P}(e_j-E_js)} \qquad .....(8)$$

and the contour L is the line from $c-i\infty$ to $c+i\infty$, suitably intended to keep poles of $\Gamma(f_j-F_jS),\ j=1,2,...,M$ to the right of the path and the singularities of $\{\Gamma(1-e_j+E_js)\}^{a_j},\ j=1,2,...N$ to the left of the path. The following sufficient conditions for the absolute convergence of the defining integral for the $\overline{H}$-function given by (7) have been given by Buschman and Srivastava [7]

$$T=\sum_{j=1}^{M}F_j+\sum_{j=1}^{N}\left|a_jE_j\right|-\sum_{j=M+1}^{Q}\left|b_jF_j\right|-\sum_{j=N+1}^{P}E_j>0, \qquad .....(9)$$

and $$\left|\arg(z)\right|<\frac{1}{2}\pi T \qquad .....(10)$$

A generalized polynomial set is defined by the following Rodrigues type formula [12, p, 64, eq. (2, 1.8)] given below :

$$S_N^{A,\beta,\tau}\left[z,r,h,q,A',B',m,k,l\right]=\left[A'x+B'\right]^{-A}\left(1-\tau x^r\right)^{-B/\tau}$$

$$T_{k,l}^{m+n}\left[\left(A'x+B'\right)^{A+qn}\left(1-\tau x^r\right)^{\frac{B}{\tau}+hn}\right], \qquad .....(11)$$

where the differential operator $T_{k,l}$ being defined as

$$T_{k,l}=x^l\left(k+x\frac{d}{dx}\right) \qquad .....(12)$$

The explicit form of this generalized polynomial set [12, p, 71, eq. (2.3, 4)] is given by

$$S_n^{A,\beta,\tau}[x,r,h,q,A',B',m,k,l]=B'^{qn}x^{l(m+n)}(1-\tau x^r)^{hn}l^{(m+n)}$$

$$\sum_{p=0}^{m+n}\sum_{e=0}^{P}\sum_{\delta=0}^{m+n}\sum_{i=0}^{\delta}\frac{(-1)^{\delta}(-\delta)_i(A)_{\delta}(-p)_e}{P!\,\delta!\,i!\,e!}\frac{(-A-qn)_i}{(1-A-\delta)!}\left(-\frac{B}{\tau}-hn\right)_p$$

$$\left(\frac{i+k+re}{l}\right)_{m+n}\left(\frac{-\tau x^r}{1-\tau x^r}\right)^p\left(\frac{A'x}{B'}\right)^i \qquad .....(13)$$

It is to be noted that the polynomial set defined by (11) is very general in nature and it unifies and extends a number of classical polynomials introduced and studied by various research workers such as Chatterjee[15], Gould-Hooper [13], Singh and Srivastava [14], etc., some of the special cases of (11) are given by Raizada in a tabular form [12]. We shall require the following explicit form of (11), which will be obtained by taking $A' = 1, B' = 0$ and let $\tau \to 0$ in (11) and use the well known confluence principle $\left[\lim_{|b| \to \infty} = (b)_n \left(\frac{x}{b} \right)^n = x^n \right]$,

we arrive at the following polynomial set

$$S_n^{A,B,0}[x] = S_n^{A,B,0}[x; r, q, 1, 0, m, k, l] = x^{qn+l(m+n)} l^{m+n}$$

$$\sum_{p=0}^{m+n} \sum_{e=0}^{P} \frac{(-p)_e}{p!e!} \left(\frac{A + qn + k + re}{l} \right)_{m+n} (Bx^r)^P \qquad \text{.....(14)}$$

2. MAIN RESULTS

This section starts with the assumption of two theorems on the product of the $\overline{H}$-function the generalized polynomial set and generalized Mittag-Leffler function associated with Saigo-Maeda fractional integral operators (1) and (2).

Theorem-2.1

Let $\alpha, \alpha', \beta, \beta', \gamma, \rho \in C, x > 0, T > 0$ and $|arg(z)| < \frac{1}{2}\pi T$ be such that

The Re $(\gamma) > 0$, $\operatorname{Re}[\rho + \lambda qn + \lambda l(m+n) + \lambda rp] > \max$

$[0, \operatorname{Re}(\alpha + \alpha' + \beta - \gamma, \operatorname{Re}(\alpha' - \beta'))]$, then there holds the formula

$$\left(I_{0,+}^{\alpha,\alpha',\beta,\beta'} Z^{\rho-1} S_n^{A,B,0}[Z^\lambda; r, q, 1, 0, m, k, l] \right.$$

$$\left. \overline{H}_{p,r}^{M,N} \left[Z^\xi \left| \begin{matrix} (e_j, E_j, a_j)_{1,N}\ (e_j, E_j)_{N+1,P} \\ (f_j, F_j)_{1,M}, (f_j, F_j, b_j)_{M,H,Q} \end{matrix} \right. \right] E_{\alpha,\beta}^{\lambda}(x) \right)(x)$$

$$= \frac{1}{\Gamma(\gamma)} x^{R-\alpha-\alpha'+\gamma-1} l^{m+n} \sum_{P=0}^{m+n} \sum_{e=0}^{P} \frac{(-p)_e}{p!e!} \left(\frac{A + qn + k + re}{l} \right)_{m+n} (B)^P$$

$$\overline{H}^{M,N+3}_{P+3,Q+3}\left[x^{\xi}\left|\begin{matrix}(1-R,\xi;1),(1-R-\gamma+\alpha+\alpha'+\beta,\xi;1),(1-R-\beta'+\alpha',\xi;1),\\ (f_j,F_j)_{1,M},(f_j,F_j;b_j)_{M+1,Q},(1-R-\beta',\xi;1),\end{matrix}\right.\right.$$

$$\left.\begin{matrix}(e_j,E_j;a_j)_{1,N}\,(e_j,E_j)_{N+1,P}\\ (1-R-\gamma+\alpha+\alpha',\xi;1),(1-R-\gamma+\alpha'+\beta,\xi;1)\end{matrix}\right]$$

$${}_1\psi_2\left[z\left|\begin{matrix}(\gamma,1)\\ (\alpha,\beta)\,(1,1)\end{matrix}\right.\right] \quad .....(15)$$

where, $R=\rho+\lambda qn+\lambda l(m+n)+\lambda rp$

Proof – 1

Making use of equations (6),(7) and(14) in left hand side of equation (15) and also using eq.(1), then interchanging the order of integrations and summations under the conditions of Theorem -1, after simplification applying the known result (3),we at once arrive at the desired result of (15).

Theorem 2.2 : Let $\alpha,\alpha^1,\beta,\beta^1,\gamma,\rho\in c$ and $x>0, T>0$ and $|\arg(Z)|<\frac{1}{2}\pi T$, be such that $\mathrm{Re}(\gamma)>0$, $\mathrm{Re}(\rho+\lambda qn+\lambda l(m+n)+\lambda rp-\xi s)<1+\min$ $[Re(\alpha+\alpha'-\gamma, Re(\alpha+\beta'-\gamma, Re(-\beta)]$ then there holds the formula :

$$\left(I^{\alpha,\alpha^1\beta,\beta^1,\gamma}_{0,-}z^{\rho-1}S^{A,B,0}_{n}\left[Z^{\lambda};r,q,1,0,m,k,l\right]\right.$$

$$\left.\overline{H}^{M,N}_{P,Q}\left[\frac{1}{z^{\xi}}\left|\begin{matrix}(e_j,E_j;a_j)_{1,N},(e_j,E_j)_{N+1,P}\\ (f_j,F_j)_{1,M},(f_j,F_j;b_j)_{M+1,Q}\end{matrix}\right.\right]E^{\gamma}_{\alpha,\beta}(z)\right)(x)$$

$$=\frac{1}{\Gamma(\gamma)}x^{R-\alpha-\alpha^1+\gamma-1}l^{m+n}\sum_{P=0}^{m+n}\sum_{e=0}^{P}\frac{(-P)_e}{P!e!}\left(\frac{A+qn+K+re}{e}\right)_{m+n}(B)^P$$

$$\overline{H}_{P+3,Q+3}^{M,N+3}\left[\frac{1}{x^{\xi}}\left|\begin{matrix}(R+\gamma-\alpha-\alpha',\xi;1),(R-\alpha-\beta'+\gamma,\xi;1),\\ (f_j,F_j)_{1,M},(f_j,F_j,b_j)_{M+1,Q}(R,\xi;1),\end{matrix}\right.\right.$$

$$\left.\begin{matrix}(R+\beta,\xi;1),(e_j,E_j,a_j)_{1,N},(e_j,E_j)_{N+1,P}\\ (R-\alpha-\alpha'-\beta'+\gamma,\xi;1),(R-\alpha+\beta,\xi;1)\end{matrix}\right]{}_1\psi_2\left[z\left|\begin{matrix}(\gamma,1)\\ (\alpha,\beta)\,(1,1)\end{matrix}\right.\right] \quad .....(16)$$

where $R=\rho+\lambda qn+\lambda l(m+n)+\lambda rp$

Proof – 2

Making use of equations (6),(7)&(14) in left hand side of eq.(16) and also using (1) ,then changing the order of integration and summations under the conditions stated in Theorem-2 and after simplification applying the known result (4) we atonce arrived at the desired result of (16).

3. SPECIAL CASES :

1. If we reduce Mittag-Leffler function $E_{\alpha,\beta}^{\gamma}(z)$ to unity in Theorem-1 & 2 then we get the known result of [10, eqs(14),(16) , pp.(138-139)] .

2. If we reduce Mittag-Leffler function and generalized polynomial set to unity and the $\overline{H}$ -function to the generalized Wright hypergeometric function,for this replacing in (15) we get the known result of [10,eq(18), pp. (140)] .

3. If we reduce $S_n^{A,B,0}$ polynomial and generalized Mittag Leffler function to unity, the $\overline{H}$ -function to Mittag-Leffler function defined by Erdelyi et.al. [3] in (15) the we obtain the known result of [10, eq.(19), pp. (140)] .

4. If we reduce the $S_n^{A,B,0}$ polynomial & generalized Mittag-Leffler function to unity in (15) and $\overline{H}$ -function to the generalized Riemann Zeta function [3, p; 27, 1.11 eq. (1)] then we get the known result of [10, eq(24), pp. (141)] .

4. CONCLUSION

The Marichev-Saigo-maeda fractional integral operator is useful for various disciplines of applied science & engineering .The importance of Marichev- Saigo –maeda functions is seen in physics, stochastic system, dynamical systems theory is steadily increasing .This operator also focuses in statiscal mechanics such as in entropy producton , reaction–diffusion etc. During the last two decades, it is been used in solving biological, physical and earth sciences.

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