ON THE NON-HOMOGENEOUS QUINTIC EQUATION WITH FIVE UNKNOWNS

$$x^3 + y^3 = z^3 + w^3 + 6T^5$$

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ABSTRACT

The non-homogeneous quintic equation with five unknowns represented by $x^3 + y^3 = z^3 + w^3 + 6T^5$ *is considered.Three different patterns of infinitely many non-zero integral solutions of the above the quintic equation are presented. Various interesting realations between the solutions and special number patterns,namely, Polygonal numbers, Star numbers, Centered Polygonal, Jacobsthal lucas numbers and Jacobsthal numbers are exhibited.*

KEYWORDS: Quintic equation with five unknowns, integral solutions.

MSC 2000 Mathematics subject classification: 11D41.

NOTATIONS

$t_{m,n}$	:	Polygonal number of rank n with size m
S_n	:	Star number of rank n
$Ct_{m,n}$	:	Centered Polygonal number of rank n with size m
j_n	:	Jacobsthal lucas number of rank n
J_n	:	Jacobsthal number of rank n

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1. INTRODUCTION

The theory of diophantine equations offers a rich variety of fascinating problems. In particular,quintic equations, homogeneous and non-homogeneous have aroused the interest of numerous mathematicians since antiquity[1-3].For illustration, one may refer [4-5] for quintic equation with three unknowns and [6-7] for quintic equation with five unknowns. This paper concerns with the problem of determining non-trivial integral solutions of the non-homogeneous quintic equation with five unknowns given by $x^3+y^3=z^3+w^3+6T^5$.A few relations among the solutions are presented.

2.METHOD OF ANALYSIS

The quintic equation with five unknowns to be solved is

$$x^3+y^3=z^3+w^3+6T^5 \tag{1}$$

To strat with,it is observed that (1) is satisfied by the following integer quintuples (x,y,z,w,T)

$$\left(2\beta+5\beta^2, 2\beta-5\beta^2, 2\beta+3\beta^2, 2\beta-3\beta^2, 2\beta\right)$$

$$\left(\begin{array}{l}(k^2+1)^n(2+5(k^2+1)^n, (k^2+1)^n(2-5(k^2+1)^n, (k^2+1)^n(2+3(k^2+1)^n,\\ (k^2+1)^n(2-3(k^2+1)^n, 2(k^2+1)^n\end{array}\right)$$

$$\left(\begin{array}{l}(k^2+1)^{2n}(8+260(k^2+1)^{2n}, (k^2+1)^{2n}(8-260(k^2+1)^{2n}, (k^2+1)^{2n}(8+252(k^2+1)^{2n},\\ (k^2+1)^{2n}(8-252(k^2+1)^{2n}, 8(k^2+1)^{2n}\end{array}\right)$$

$$\left(\begin{array}{l}2k\beta+\beta^2(4k^{4n}+1), 2k\beta-\beta^2(4k^{4n}+1), 2k\beta+\beta^2(4k^{4n}-1),\\ 2k\beta-\beta^2(4k^{4n}-1), 2k\beta\end{array}\right)$$

However, we have other patterns of solution to (1) which are presented below.

2.1: Pattern: 1

Introducing the transformations,

$$\left.\begin{array}{l}x=(k^2+1)^n c+1\\ y=(k^2+1)^n c-1\\ z=a-1\\ w=a+1\\ T=\alpha^2\end{array}\right\} \dots\dots\dots \tag{2}$$

in (1),it leads to

$$\left((k^2+1)^n c\right)^2=a^2+(\alpha^5)^2 \tag{3}$$

This is in the form of pythogorean equation satisfied by

$$a=R^2-S^2, (k^2+1)^n c=R^2+S^2, \alpha^5=2RS, \ R>S>0 \tag{4}$$

Choosing $R=(k^2+1)^n\overline{R}, S=(k^2+1)^n\overline{S}$ (5)
in (4),we get

$$\left.\begin{aligned} a&=(k^2+1)^n(\overline{R}^2-\overline{S}^2)\\ c&=(k^2+1)^n(\overline{R}^2+\overline{S}^2)\end{aligned}\right\} \quad (6)$$

$$\alpha^5=2(k^2+1)^n\overline{R}\overline{S} \quad (7)$$

It is to be noted that $\overline{R}$ and $\overline{S}$ should be chosen such that $\overline{R}\overline{S}=16(k^2+1)^{3n}\beta^5$ and thus from (7) we have

$$\alpha=2(k^2+1)^n\beta \quad (8)$$

Thus substituting the values of $\overline{R}$ and $\overline{S}$ in (6) and using (8) and (2),the non-zero distinct integral solutions to (1) are obtained.

2.2: Illustration:

Take
$$\overline{R}=8(k^2+1)^n\beta^4$$
$$\overline{S}=2(k^2+1)^n\beta$$

Therefore,

$$a=(k^2+1)^{4n}(4\beta^2)\left(16(k^2+1)^{2n}\beta^6-1\right)$$
$$c=(k^2+1)^{4n}(4\beta^2)\left(16(k^2+1)^{2n}\beta^6+1\right)$$

Hence the corresponding integral solutions to (1) are represented by

$$x=\left[(k^2+1)^{4n}(4\beta^2)\left(16(k^2+1)^{2n}\beta^6+1\right)\right]+1$$
$$y=\left[(k^2+1)^{4n}(4\beta^2)\left(16(k^2+1)^{2n}\beta^6-1\right)\right]+1$$
$$z=\left[(k^2+1)^{4n}(4\beta^2)\left(16(k^2+1)^{2n}\beta^6+1\right)\right]-1$$
$$w=\left[(k^2+1)^{4n}(4\beta^2)\left(16(k^2+1)^{2n}\beta^6-1\right)\right]-1$$
$$T=4(k^2+1)^{2n}\beta^2$$

2.3: Properties:

1. Each of the following is a perfect square
(a) $xy+x-y$
(b) $xw-x-w+2$
2. $T-(k^2+1)^{2n}t_{10,\beta}\equiv 0(\text{mod}3)$
3. $6\left(\frac{w-1}{T^3}+\frac{(k^2+1)^{2n}}{T^2}\right)$ is a nasty number.
4. For odd values of β, $\left(\frac{w-1}{T^3}+\frac{(k^2+1)^{2n}}{T^2}\right)-8t_{3,s}=1$

5. $6\left(\frac{w-1}{T^3}+\frac{(k^2+1)^{2n}}{T^2}\right)-S_\beta+1\equiv 0 \pmod 6$

6. $\left(\frac{z+1}{T^3}-\frac{(k^2+1)^{2n}}{T^2}\right)-Ct_{2,\beta}-t_{6,\beta}\equiv -1 \pmod 2$

7. For values of $\beta=2^{2s}$, we have

(a) $\left(\frac{x-1}{T^3}-\frac{(k^2+1)^{2n}}{T^2}\right)-3J_{4,S}=1$

(b) $\left(\frac{y+1}{T^3}+\frac{(k^2+1)^{2n}}{T^2}\right)-j_{4,s}+1=0$

2.4: Pattern: 2

It is worth to mention that (3) is also satisfied by

$$(k^2+1)^n c = R^2+S^2, a=2RS \tag{9}$$

$$\alpha^5 = R^2-S^2 \tag{10}$$

Choosing

$$R=(k^2+1)^n \overline{R}, \quad S=(k^2+1)^n \overline{S}, \quad \overline{R}>\overline{S}$$

in (11),we get

$$\alpha^5=(k^2+1)^{2n}(\overline{R}^2-\overline{S}^2) \tag{11}$$

Taking

$$\overline{R}^2-\overline{S}^2=(k^2+1)^{3n}(2\beta)^5 \tag{12}$$

We have

$$\alpha=(k^2+1)(2\beta) \tag{13}$$

On solving (12) the values of $\overline{R}$ and $\overline{S}$ are given by

$$\overline{R}=\frac{1}{2}\left((k^2+1)^{2n}(2\beta)^4+(k^2+1)^n(2\beta)\right)$$

$$\overline{S}=\frac{1}{2}\left((k^2+1)^{2n}(2\beta)^4-(k^2+1)^n(2\beta)\right)$$

Thus substituting the values of $\overline{R}$ and $\overline{S}$ in (9) and using (13) and (2), the corresponding integral solutions to (1) are given by

$$x=\left[(k^2+1)^{4n}(2\beta^2)\left((k^2+1)^{2n}(2\beta)^6+1\right)\right]+1$$

$$y=\left[(k^2+1)^{4n}(2\beta^2)\left((k^2+1)^{2n}(2\beta)^6-1\right)\right]+1$$

$$z=\left[(k^2+1)^{4n}(2\beta^2)\left((k^2+1)^{2n}(2\beta)^6+1\right)\right]-1$$

$$w=\left[(k^2+1)^{4n}(2\beta^2)\left((k^2+1)^{2n}(2\beta)^6-1\right)\right]-1$$

$$t=(k^2+1)^2(2\beta)^2$$

2.5: Pattern:3

Introducing the transformations,

$$\left.\begin{aligned} x &= (k^2+1)^n c + 1 \\ y &= (k^2+1)^m a - 1 \\ z &= (k^2+1)^n c - 1 \\ w &= (k^2+1)^m a + 1 \\ t &= \alpha^2 \end{aligned}\right\} \quad \text{..........(14)}$$

in (1),it leads to

$$\left((k^2+1)^n c\right)^2 = \left((k^2+1)^m a\right)^2 + (\alpha^5)^2 \quad (15)$$

which is again in the form of well known pythogorean equation satisfied by

$$\left.\begin{aligned} (k^2+1)^n c &= p^2 + q^2 \\ (k^2+1)^m a &= p^2 - q^2 \end{aligned}\right\} \quad \text{..........(16)}$$

$$\alpha^5 = 2pq \quad (17)$$

Choosing

$$\begin{aligned} p &= 4(k^2+1)^{m+n}\overline{P} \\ q &= 4(k^2+1)^{m+n}\overline{Q} \end{aligned} \quad , \overline{P} > \overline{Q} \quad (18)$$

in (17) ,we get

$$\alpha^5 = 2(16)(k^2+1)^{2m+2n}\overline{P}\,\overline{Q} \quad (19)$$

Again,taking

$$\begin{aligned} \overline{P} &= (k^2+1)^{3m} S^5 \\ \overline{Q} &= (k^2+1)^{3n} T^5 \end{aligned} \quad , S,T \neq 0 \quad (20)$$

in (19),we have

$$\alpha = 2(k^2+1)^{m+n} ST \quad (21)$$

Thus, substituting the values of p and q in (16) and using (21) in (14),the corresponding integral solutions to (1) are as follows:

$$x = \left[16(k^2+1)^{8m+2n}(S^{10}+T^{10})\right] + 1$$

$$y = \left[16(k^2+1)^{8m+2n}(S^{10}-T^{10})\right] - 1$$

$$z = \left[16(k^2+1)^{8m+2n}(S^{10}+T^{10})\right] - 1$$

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$$w = [16(k^2+1)^{8m+2n}(S^{10}-T^{10})]+1$$

$$t = 4(k^2+1)^{2m+2n}(ST)^2$$

3.Conclusion:

It is worth to mention here that the solution of (16) may also be considered as $(k^2+1)^m a = 2pq, \alpha^5 = p^2 - q^2$.Proceeding as in pattern 3.one may obtain non-zero integral solutions to (1). To conclude, one may search for other pattern of solutions and their corresponding properties for the quintic equation under consideration.

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