TWO WAYS OF APPROXIMATION OF PI AND ITS SQUARE ROOT

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Abstract

In this paper we discuss two methods of approximations of π and it's square root. The approximation of π in the first method uses a unit circle and has been around for quite some time, but the part concerning approximation of $\sqrt{\pi}$ is new. The second approximation makes use of the bell shaped curve $y = e^{-x^2}$ and the error function. Both the approximations are based on the Monte Carlo Simulation Method. The method requires of picking N points (called sample) at random and counting those that satisfy certain criteria. The number N is called the sample size. Here we determine the minimum sample sizes required to obtain an approximation so that the absolute error is less than a given value.

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1. Introduction

In this document, we describe methods that can be successfully implemented to achieve rational approximations of π and $\sqrt{\pi}$ at any desired level of accuracy. Both the approximations are based on the Monte Carlo Simulation Method. It consists of picking points at random and counting those that satisfy certain criteria. The number of points that are picked is called the sample size of the simulation. We state below a theorem that will be used later to find this sample size.

Theorem (1.1) Let S be a region on R^2 and U be a region contained in S i.e. $U \subset S$, and let p_{ex} denote the probability that a point chosen randomly from S lies in U. Suppose we are conducting an experiment of picking N points randomly from S. Let N_U be the number of points from S that lie in U. Then for any error level r, the proportion $p_{es} = \frac{N_U}{N}$ approximates p_{ex} with an assurance of 99% if $N = \frac{1.65}{r^2}$ or greater.

Proof. Note that we are here concerned with a binomial experiment and our sample comes from a population where the proportion of success (i.e., the probability that the point chosen comes from U) is p_{es}. Then by the Theorem on Sampling Distribution of the Sample Proportion [2], the sampling distribution of p_{es} has mean p_{ex} and standard error $\sigma = [\frac{p_{ex}(1-p_{ex})}{N}]^{\frac{1}{2}}$. Moreover the standardized variable $Z = \frac{p_{es} - p_{ex}}{\sigma}$ is asymptotically normal. Below we show how this fact can be used to approximate p_{ex} by p_{es}.

Suppose we want p_{es} to approximate p_{ex} with an assurance of $k\%$ at an error level of r. Then $k = P(|p_{es} - p_{ex}| \le r)$. Note that

$$P(|p_{es} - p_{ex}| \le r) = P(|Z\sigma| \le r) = P(|Z| \le \frac{r}{\sigma}) = 2P(0 < Z \le \frac{r}{\sigma}) = 2[P(Z \le \frac{r}{\sigma}) - \frac{1}{2}] =$$

$= 2P(Z \leq \frac{r}{\sigma}) - 1$. Then $P(Z \leq \frac{r}{\sigma}) = \frac{1+k}{2}$. For k we take $.99$ and obtain $P(Z \leq \frac{r}{\sigma}) = .995$. By [3], we get $\frac{r}{\sigma} = 2.57$ i.e., to achieve an assurance of 99% such that $|p_{es} - p_{ex}| \leq r$, we have to have $r^2 = 6.6\sigma^2$. As $\sigma^2 = \frac{p_{ex}(1-p_{ex})}{N}$, we get $r^2 = 6.6\frac{p_{ex}(1-p_{ex})}{N}$ and therefore $N = \frac{6.6}{r^2} p_{ex}(1-p_{ex})$. As the error level r and sample size N are inversely related, by increasing the sample size we can reduce the error level as low as we wish i.e., p_{es} can be made to approximate p_{ex} as close as one desires. Consequently, any sample of size $> N = \frac{6.6}{r^2} p_{ex}(1-p_{ex})$, will make the inequality $|p_{es} - p_{ex}| \leq r$ hold. Note that $f(x) = x(1-x)$ attains an absolute maximum of $\frac{1}{4}$ at $x = \frac{1}{2}$ on the interval $[0,1]$. So for our sample we will choose a size of $N = \frac{6.6}{r^2} \times \frac{1}{4} = \frac{1.65}{r^2}$. For such a sample, $|p_{es} - p_{ex}| \leq r$ holds. ■

2. The First Approximation

In this method π is approximated first and then approximation of $\sqrt{\pi}$ is initiated. To approximate π, a number of points say N is randomly chosen from a square of side length 2 with a unit circular disk U inside. If N_U is the number of picked points that lie in U, then proportion $\frac{N_U}{N}$ approximates $\frac{\pi}{4}$ and hence $\frac{4N_U}{N}$ is an approximation of π. This idea has in fact been around for some time and it is to be found in [1]. There the author after a brief introduction of the method above proceeds to write a java code with $N = 1000$ which delivers an estimate of π. He does not discuss how the sample size N is to be chosen given an error level e. Below we produce results that should answer to this question.

Theorem(2.1) Suppose we conduct an experiment of picking N points randomly from a square $S = [-1,1] \times [-1,1]$. Let N_U be the number of those picked points from N that lie in

$U = \{(x, y) \in S \mid x^2 + y^2 \le 1\}$, and let $p_{es} = \frac{N_U}{N}$. Then given an error level e, if $N = \frac{26.4}{e^2}$ then there is $.99$ chance that $\overline{\pi} = 4p_{es} = \frac{4N_U}{N}$ approximates π such that $|\overline{\pi} - \pi| \le e$.

Proof. Let p_{ex} denote the probability that a point chosen randomly from S lies in U. Hence $p_{ex} = \frac{\pi(1)^2}{4} = \frac{\pi}{4}$ implying $\pi = 4p_{ex}$. We set $r = \frac{e}{4}$ and use Theorem(2.1) to find a p_{es} such that $|p_{es} - p_{ex}| \le r$. Then $|\pi - \overline{\pi}| = |4p_{ex} - 4p_{es}| \le 4r = 4 \cdot \frac{e}{4} = e$. Hence again by Theorem(2.1), a sample size of $\frac{1.65}{r^2} = \frac{1.65}{(\frac{e}{4})^2} = \frac{26.4}{e^2}$ or greater may be used for approximation of π by $\overline{\pi}$.■

We now proceed to approximate $\sqrt{\pi}$.

Theorem (2.2). Suppose we conduct an experiment of picking N points randomly from a square $S = [0, \pi] \times [0, \pi]$. Let N_U be the number of those points from N that lie in $U = \{(x, y) \in S \mid y \le \sqrt{x}\}$, and let $p_{es} = \frac{N_U}{N}$. Given an error level e, there is an assurance of 99% that if a sample size of $N = \frac{36.67}{e^2(1-e)^2}$ is chosen for our experiment then $\left|\sqrt{\pi} - \overline{\pi}\right| \le e$ where $\overline{\pi} = \frac{2}{3p_{es}} = \frac{2e(1-e)}{3\pi}$.

Proof. By Theorem(1.1), a sample size of $N = \frac{1.65}{r^2}$ yields $p_{es} = \frac{N_U}{N}$ satisfying $\left|p_{es} - p_{ex}\right| \le r$. Hence $p_{es} > p_{ex} - r$ or equivalently $p_{es}\, p_{ex} > p_{ex}(p_{ex} - r)$.

Set $r = \frac{2(e - e^2)}{3\pi}$. Then

$p_{ex}(p_{ex} - r)$

$$= \frac{2}{3\sqrt{\pi}}\left(\frac{2}{3\sqrt{\pi}} - r\right) \text{ as } p_{ex} = \frac{\int_0^{\pi} \sqrt{x}dx}{\pi^2} = \frac{2}{3\sqrt{\pi}}$$

$$= \frac{4}{9\pi} - \frac{2r}{3\sqrt{\pi}}$$

$$= \frac{4-4e}{9\pi} + \frac{4e}{9\pi} - \frac{2r}{3\sqrt{\pi}}$$

$$= \frac{4-4e}{9\pi} + \frac{1}{9\pi}(4e - 6r\sqrt{\pi})$$

$$= \frac{4-4e}{9\pi} + \frac{1}{9\pi}(4e - 6 \times \frac{2(e-e^2)}{3\pi} \times \sqrt{\pi})$$

$$= \frac{4-4e}{9\pi} + \frac{1}{9\pi}(4e - \frac{4(e-e^2)}{\sqrt{\pi}})$$

$$= \frac{4-4e}{9\pi} + \frac{1}{9\pi}(4e - \frac{4e}{\sqrt{\pi}} + \frac{4e^2}{\sqrt{\pi}})$$

$$= \frac{4-4e}{9\pi} + \frac{4e^2}{9\pi\sqrt{\pi}} + \frac{4e}{9\pi}(1 - \frac{1}{\sqrt{\pi}}) .$$

Hence $p_{es}\, p_{ex} > p_{ex}(p_{ex} - r) > \frac{4(1-e)}{9\pi}$ and therefore

$$\left|\sqrt{\pi} - \sqrt{\pi}\right|$$

$$= \left|\frac{2}{3p_{ex}} - \frac{2}{3p_{es}}\right|$$

$$= \frac{2}{3}\left|\frac{1}{p_{ex}} - \frac{1}{p_{es}}\right|$$

$$= \frac{2}{3}\frac{\left|p_{es} - p_{ex}\right|}{p_{es}\, p_{ex}}$$

$$\leq \frac{2}{3}\frac{r}{p_{es}\, p_{ex}}$$

$$= \frac{2}{3} \frac{\frac{2e(1-e)}{3\pi}}{p_{es} p_{ex}}$$

$$< \frac{2}{3} \frac{\frac{2e(1-e)}{3\pi}}{\frac{4(1-e)}{9\pi}}$$

$= e$.

Next we proceed to compute the sample size which is

$$N = \frac{1.65}{r^2} = \frac{1.65}{[(\frac{2(e-e^2)}{3\pi})]^2} = \frac{1.65 \times 9\pi^2}{4e^2(1-e)^2} = \frac{36.67}{e^2(1-e)^2}. \blacksquare$$

As e is usually equal or less than 0.05, $(1-e)^2 \geq (1-0.05)^2 = (0.95)^2 = 0.90$ and therefore $\frac{36.67}{e^2(1-e)^2} \leq \frac{36.67}{e^2(.90)^2} = \frac{45.27}{e^2}$. Hence for two decimal approximation or more, it will suffice to take $N = \frac{46}{e^2}$.

3. The Second Approximation

Contrary to the first approximation, here we first find an approximation of $\sqrt{\pi}$ and then proceed to approximate π. We begin with a theorem.

Theorem(3.1). Suppose we conduct an experiment of picking N points randomly from a rectangle $S = [0,5] \times [0,1]$. Let N_U be the number of those points from N that lie in $U = \{(x, y) \in S \mid y \leq e^{-x^2}\}$, and let $p_{es} = \frac{N_U}{N}$. Then given an error level e, if $N = \frac{165}{e^2}$ then there is $.99$ chance that $w = 10p_{es} = \frac{10N_U}{N}$ approximates $\sqrt{\pi}$ such that $| w - \sqrt{\pi} | \leq e$.

Proof. Let p_{ex} denote the probability that a point chosen randomly from S lies in U.

Hence $p_{ex} = \frac{\int_0^5 e^{-x^2} dx}{5} = \frac{\frac{1}{2} erf(5)\sqrt{\pi}}{5} = \frac{\sqrt{\pi}}{10}$ as $erf(5) = 1$. Thus $\sqrt{\pi} = 10 p_{ex}$.

Recall that $w = 10 p_{es} = \frac{10 N_U}{N}$. Hence

$\left|\sqrt{\pi} - w\right| = \left|10 p_{ex} - 10 p_{es}\right| = 10\left|p_{ex} - p_{es}\right| \le 10r$ by Theorm (1.1) if $N = \frac{1.65}{r^2}$. Set $r = \frac{e}{10}$.

Then $\left|\sqrt{\pi} - w\right| \le 10r = 10 \times \frac{e}{10} = e$ and $N = \frac{1.65}{r^2} = \frac{1.65}{(\frac{e}{10})^2} = \frac{165}{e^2}$. So for any $e > 0$, $\left|\sqrt{\pi} - w\right| \le e$ if $N = \frac{165}{e^2}$. ■

Next we proceed to approximate π and find the corresponding sample size of approximation.

Since $\sqrt{\pi} = 10 p_{ex}$, we have $\pi = 100 p_{ex}^2$. Let $\bar{p} = 100 p_{es}^2$. Then $\left|\pi - \bar{p}\right| = \left|100 p_{ex}^2 - 100 p_{es}^2\right| = 100(p_{ex} + p_{es})\left|p_{ex} - p_{es}\right| < 100 \times 2 \times \left|p_{es} - p_{ex}\right| < 200r$. Set $r = \frac{e}{200}$.

Then $\left|\pi - \bar{p}\right| < 200r = 200 \times \frac{e}{200}$ and $N = \frac{165}{r^2} = \frac{165}{(\frac{e}{200})^2} = \frac{66000}{e^2}$. Thus we have the following theorem.

Theorem(3.2) Suppose we conduct an experiment of picking N points randomly from a rectangle $S = [0,5] \times [0,1]$. Let N_U be the number of those points from N that lie in $U = \{(x, y) \in S \mid y \le e^{-x^2}\}$, and let $p_{es} = \frac{N_U}{N}$. Then given an error level e, if $N = \frac{66000}{e^2}$ then there is .99 chance that $\bar{p} = 100 p_{es}^2$ approximates π such that $|\bar{p} - \pi| \le e$.

References

[1] John R. Hubbard, "Programming with C^{++}", Schaum's International Edition, pp 82, McGraw –Hill, Singapore (1996).

[2] Paul Newbold, Statistics for Business & Economics, 4th ed., pp 234-235, Prentice-Hall, Inc.,Englewood Cliffs, New Jersey (1995).

[3] Paul Newbold, Statistics for Business & Economics, 4th ed., pp 836, Prentice-Hall, Inc.,Englewood Cliffs, New Jersey (1995).