A COMMON FIXED POINT THEOREM FOR SIX SELF MAPPING UNDER S-B PROPERTY

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Abstract

In this paper we use the S-B property with semi compatibility, weakly compatibility and prove common fixed point theorem for six self mapping on a fuzzy metric space.

Keywords- Weakly compatible, semi compatible, (S-B) property, Fuzzy metric space

Zadeh [15] was the foundation of fuzzy sets in 1965. To use this concept in topology and analysis, many authors have extensively developed the theory of fuzzy sets and its applications. Especially, Deng [2], Erceg [3], and Kramosil and Michalek [8] have introduced the concepts of fuzzy metric spaces in different ways. In 1988, Grabiec [7] extended the fixed point theorem in fuzzy metric space. George and Veeramani [5], [6] have modified the concept of fuzzy metric space introduced by Kramosil and Michalek. They have also shown that every metric induces a fuzzy metric. Singh et al. [14] proved various fixed point theorems using the concepts of semi-compatibility, compatibility in Fuzzy metric space.

In this paper, we proved common fixed point theorem for six self mapping by using the semi compatibility, weakly compatibility and new (S-B) property in fuzzy metric space, and using the result from [9] and [11]

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Preliminaries

Definition1. [15] Let X be any set**.** A fuzzy set in X is a function with domain X and values in [0, 1].

Definition2. [11] A binary operation $*$: [0, 1] × [0, 1] → [0, 1] is said to be a continuous t-norm if $*$ satisfies the following conditions: For a, b, c, d ∈ [0, 1],

(i) $*$ is associative and commutative,

(ii) $*$ is continuous,

(iii) $a * 1 = a$, for all $a \in [0, 1]$

(iv) $a * b \leq c * d$ whenever $a \leq$ c and $b \leq d$, for each a, b, c, d ∈ [0, 1].

We observe that $a * b = \min\{a, b\}$ is a t-norm.

Definition3. [8] The triplet (X, M, *) is said to be a **Fuzzy metric space if** is an arbitrary set, is a continuous t-norm and is a Fuzzy set on X x X x [0, ∞] → [0, 1], satisfying the following conditions, for all x, y, z ∈ X and s, t > 0.

(FM-1) M(x, y, 0) = 0,

(FM-2) M(x, y, t) = 1 for all t >0 if and only if x = y,

(FM-3) M(x, y, t) = M(y, x, t),

(FM-4) M(x, y, t) * M(y, z, s) ≤ M(x, z, t+s),

(FM-5) M(x, y, .) : [0, ∞]→ [0, 1], is left continuous,

(FM-6) $\lim_{n\to\infty}$ M(x, y, t) = 1.

Note that can be considered as the degree of nearness between x and y with respect to t. We identify x = y with M(x, y, t) = 1for all t > 0.

Definition4. [5] Let (X, M,*) be fuzzy metric space then,

a) A sequence $\{x_n\}$ in X is said to be **convergent** to x in X if for each ε > 0 and each t > 0, there exists $n_0 \in$ N such that $M(x_n, x, t) > 1- \varepsilon$ for all $n \geq n_0$.

b) A sequence $\{x_n\}$ in X is said to be **Cauchy sequence** for each ε > 0 and t > 0, there exists $n_0 \in$ N such that $M(x_m, x_n, t) > 1- \varepsilon$ for all $m, n \geq n_0$.

c) A fuzzy metric space in which every Cauchy sequence is convergent is said to be **complete**.

Definition5. [9] Self mappings A and S of a Fuzzy metric space (X, M, *) are said to be **compatible** if and only if $M(ASx_n, SAx_n, t) \to 1$ for all $t > 0$, whenever $\{x_n\}$ is a sequence in X such that $Sx_n, Ax_n \to p$ for some p in X as $n \to \infty$.

Definition6. [14] Two self mapping A and S of a Fuzzy metric space(X, d) are said to be **weakly compatible** if they commute at their coincidence points i.e., if Ax = Sx then ASx = SAx.

Definition7. [14] Two self mapping A and S of a Fuzzy metric space(X, d) are said to be **semi-compatible** if $\lim_{n\to\infty} d(ASx_n, Sx_n, a) = 0$ for all $a \in X$, where $\{x_n\}$ is a sequence in X such that if $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = x$ for some x in X.

Definition8. ([13], [12]) Let S and T be two self mappings of a fuzzy metric space$(X, M, *)$.We say that S and T satisfy the property S-B if there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = z$ for some $z \in X$.

Lemma1. [4] Let (X, M, *) be a fuzzy metric space. For all x, y ∈ X, M(x, y, •) is non decreasing.

Lemma2. [1] Let (X, M, *) be a fuzzy metric space if there exists $k \in (0, 1)$ such that $M(x, y, kt) \geq M(x, y, t)$ then x=y.

Example1. [13] Let $X = [0, +\infty)$. Define A, B: $X \to X$ by $Ax = x + 1/2$ *and* $Bx = 2x+1/2$, $x \in X$. Consider the sequence $x_n = 1/n$, clearly $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = 1/2$.Then A and B satisfy (S-B) property.

Example2. Let $X = [2, +\infty)$. Define A, B : $X \to X$ *by* $Ax = 2x + 1$ *and* $Bx = x + 1$, for all $x \in X$. Suppose property $(S-B)$ holds, then there exists a sequence $\{x_n\}$ in X satisfying

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = z \text{ for some } z \in X.$$

Therefore $\lim_{n\to\infty} x_n = z - 1$ *and* $\lim_{n\to\infty} x_n = (z-1)/2$.

Then $z = 1$, which is a contradiction since $1 \notin X$. Hence A and B do not satisfy the property (S-B).

Rajinder Sharma [12] proved the following:

Theorem 1.1 Let $(X, M, *)$ be a fuzzy metric space with $t*t \geq t$ for all $t \in [0,1]$ and condition (FM-6). Let $A, B\ S$ *and* T be mappings of X into itself such that

(1.1.1) $A(X) \subset T(X)$ *and* $B(X) \subset S(X)$,

(1.1.2) (A, S) or (B, T) satisfies the property $(S-B)$,

(1.1.3) there exists a constant $k \in (0, 1)$ such that

$M^{2p}(Ax,By,kt) \geq \min\{M^{2p}(Sx,Ty,t), M^{q}(Sx,Ax,t), M^{q'}(Ty,By,t), M^{r}(Sx,By,t), M^{r'}(Ty,Ax,(2-\alpha)t), M^{s}(Sx,Ax,t),$

$M^{s'}(Ty,Ax,(2-\alpha)t), M^{l}(Sx,By,t), M^{l'}(Ty, By,t)\}$

for all $x, \in X$, $\alpha \geq 0, \alpha \in (0,2)$, $t > 0$ and $0 < p, q, q', r, r', s, s', l, l' \leq 1$ such that

$2p = q + q' = r + r' = s + s' = l + l'$.

(1.1.4) the pairs (A, S) and (B, T) are weakly compatible,

(1.1.5) one of (X), $B(X)$, $S(X)$ or $T(X)$ is a closed subset of X, then A,B,S and T have a unique common fixed point in X.

N. Umadevi [10] proved the following:

Theorem 1.2 Let (X, M, *) be a fuzzy metric with * is min t- norm with condition (1). Let A, B, Sand T be mappings of X into itself such that

(1.2.1) A(X) ⊂ T(X), B(X) ⊂ S(X), T(X) is closed subset of X,

(1.2.2) {B, T} satisfies the property (S-B),

(1.2.3) their exist a constant k ∈ (0, 1) and α ∈ (0, 2) such that k < α, k + α < 2 and satisfies

$M(Ax, By, kt) \geq \min\{M(Sx, Ty, t)*M(Sx, Ax, t)*M(Ty, By, t)*M(Sx, By, t)*M(Ty, Ax, (2-\alpha)t\}$

for all t > 0.

(1.2.4) (A, S) and (B, T) are weakly compatible,

Then A, B, S and T have a unique common fixed point in X.

Main Result

Theorem 1.3 Let (X, M, *) be a fuzzy metric with a*a ≥ a for all a ∈ [0, 1] and the condition (fm6). Let A, B, S, T, P and Q be mappings from X into itself such that

(1.3.1) P(X) ⊂ AB(X), Q(X) ⊂ ST(X),

(1.3.2) the pairs (A, B), (S, T), (Q, B), (T, P) and are commuting mappings,

(1.3.3) the pairs (P, ST) are semi-compatible and (Q, AB) are weakly compatible,

(1.3.4) {P, ST} or {Q, AB} satisfies the property (S-B),

(1.3.6) $M(Px, Qy, qt) \geq M(STx, ABy, t) * M(Px, STx, t) * M(Qy, ABy, t) * M(Px, ABy, t)$,

(1.3.5) one of AB(X), P(X), AB(X) or Q(X) is closed subset of X,

Then A, B, S, T, P and Q have a unique common fixed point in X.

Proof: Without loss of generality we suppose that (Q, AB) satisfies the S-B property, so there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Qx_n = \lim_{n\to\infty} ABx_n = z$ for some $z \in X$.

Since $Q(X) \subset ST(X)$ there exists a sequence $\{y_n\}$ in X such that $Qx_n = STy_n$.So $\lim_{n\to\infty} STy_n = z$.

Now we prove that $\lim_{n\to\infty} Py_n = z$.

By (1.3.6) we get,

$M(Py_n, Qx_n, qt) \geq M(STy_n, ABx_n, t) * M(Py_n, STy_n, t) * M(Qx_n, AB\, x_n, t) * M(Py_n, ABx_n, t)$,

$M(Py_n, z, qt) \geq M(z, z, t) * M(Py_n, z, t) * M(z, z, t) * M(Py_n, z, t)$,

Therefore by lemma 2, $M(Py_n, z, qt) \geq M(Py_n, z, t)$

we have $Py_n = z$.

Since (P, ST) are semi compatible pairs, we have $PSTy_n = Pz$ and $STPy_n = STz$.

Since the limit in fuzzy metric space is unique, we get Pz = STz.

By (1.3.6) we get,

$M(PPy_n, Qx_n, qt) \geq M(STPy_n, ABx_n, t)*M(PPy_n, STy_n, t)*M(Qx_n, ABx_n, t)*M(PPy_n, ABx_n, t)$

Taking limit $n\to\infty$, we have

$M(Pz, z, qt) \geq M(Pz, z, t) * M(Pz, z, t) * M(z, z, t) * M(Pz, z, t)$

Therefore by lemma 2, $M(Pz, z, qt) \geq M(Pz, z, t)$.

we have Pz = STz = z.

By (1.3.6) we get,

$M(PTz, Qy_n, qt) \geq M(STTz, ABy_n, t) * M(PTz, STTz, t) * M(Qy_n, ABy_n, t) * M(PTz, ABy_n, t)$,

Since ST = TS, PT = TP, so we have

STTz = (TS)Tz = T(STz) = Tz and PTz = T(Pz) = Tz, we have

Taking limit $n\to\infty$, we have

$M(Tz, z, qt) \geq M(Tz, z, t) * M(Tz, Tz, t) * M(z, z, t) * M(Tz, z, t)$,

Therefore by lemma 2, $M(Tz, z, qt) \geq M(Tz, z, t)$

we have Tz = z. So we have Pz = z.

hence Tz = Pz = Sz = z.

Suppose AB(X) is closed subset of X. there exists $u \in X$ such that z = ABu.

By (1.3.6) we get,

$M(Py_n, Qu, qt) \geq M(STy_n, ABu, t) * M(Py_n, STx_n, t) * M(Qu, ABu, t) * M(Py_n, ABu, t)$,

Taking limit $n\to\infty$, we have

$M(z, Qu, qt) \geq M(z, z, t) * M(z, z, t) * M(Qu, z, t) * M(z, z, t)$,

Therefore By lemma 2 $M(z, Qu, qt) \geq M(Qu, z, t)$,

hence Qu = z = ABu.

Since (Q, AB) are weakly compatible, we have QABu = ABQu. Thus Qz = ABz.

By (1.3.6) we get,

$M(Py_n, Qz, qt) \geq M(STy_n, ABz, t) * M(Py_n, STy_n, t) * M(Qz, ABz, t) * M(Py_n, ABz, t)$,

Taking limit $\square \rightarrow \infty$, we have

$M(z, Qz, qt) \geq M(Qz, Qz, t) * M(z, Qz, t) * M(Qz, Qz, t) * M(z, Qz, t)$,

Therefore By lemma 2 $M(z, Qz, qt) \geq M(Qz, Qz, t)$

Thus Qz = z.

By (1.3.6) we get,

$M(Py_n, QBz, qt) \geq M(STy_n, ABBz, t) * M(Py_n, STy_n, t) * M(QB_z, ABBz, t) * M(Py_n, ABBz, t)$,

Since AB = BA and QB = BQ

ABBz = B(ABz) = Bz and QBz = BQz = Bz.

Taking limit $\square \rightarrow \infty$, we have

So $M(z, Bz, qt) \geq M(z, Bz, t) * M(z, z, t) * M(Bz, Bz, t) * M(z, Bz, t)$,

Therefore By lemma 2 $M(z, Bz, qt) \geq M(z, Bz, t)$

Thus Bz= z.

So Az = z thus Qz = Az = Bz = z.

Hence z, is the common fixed point of A, B, P, Q, S and T.

Uniqueness Let w is another common fixed point of A, B, P, Q,S and T

By (1.3.6) we get,

$M(Pz, Qw, qt) \geq M(STz, ABw, t) * M(Pz, STz, t) * M(Qw, ABw, t) * M(Pz, ABw, t)$,

$M(z, w, qt) \geq M(z, w, t) * M(z, z, t) * M(w, w, t) * M(z, w, t)$,

Therefore By lemma 2 $M(z, w, qt) \geq M(z, w, t)$

Hence z = w.

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