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HANKEL TYPE TRANSFORMATION ASSOCIATED WITH CONVOLUTION OPERATORS AND MULTIPLIERS ON HARDY TYPE SPACES

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Abstract:

In this paper we study Hankel type transformation on Hardy type spaces. Further we investigate Hankel type convolution operators and Hankel type multipliers on these Hardy type spaces.

Keywords: Hankel type transform, Hankel type multipliers, Hankel type convolution operators.

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1. Introduction and Preliminaries:

Following [29], we define the Hankel type transform as

$$h_{\alpha,\beta}(\phi)(y) = \int_0^\infty (xy)^{-(\alpha-\beta)} J_{\alpha-\beta}(xy)\, \phi(x) x^{4\alpha}\, dx,$$

where J_λ denotes the Bessel type function of the first kind and order $\alpha-\beta$. Throughout this paper we will assume that $(\alpha-\beta) \geq -\frac{1}{2}$.

For every $1 \leq p < \infty$, we consider the space $L^p_{\alpha,\beta}$ constituted by all these Lebesgue measurable functions ϕ on $(0,\infty)$ such that

$$\|\phi\|_p = \left\{\int_0^\infty |\phi(x)|^p\, d\nu(x)\right\}^{1/p} < \infty,$$

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where $dv(x)$ denotes the measure $\left(x^{4\alpha}/2^{\alpha-\beta}\Gamma(3\alpha+\beta)\right)dx$. By $L^p_{\alpha,\beta}$ we understand the space $L_\infty\left((0,\infty),\ dx\right)$ of the essentially bounded functions on $(0,\infty)$.

It is clear that $h_{\alpha,\beta}$defines a continuous mapping from $L^1_{\alpha,\beta}$ into $L^\infty_{\alpha,\beta}$. Herz [17, Theorem 3]established that $h_{\alpha,\beta}$ can be extended to $L^p_{\alpha,\beta}$ as a continuous mapping from $L^p_{\alpha,\beta}$ into $L^{p'}_{\alpha,\beta,}$ for every $1 \le p \le 2$. Here p' denotes the conjugate of $p\left(that\ is\ p' = p/(p-1)\right)$.

In [2, Lemma 3.1] we proved by using the Marcinkiewicz interpolation theorem the following $L^p -$ inequality that is a Pitt type inequality for the Hankel transformation [12, Corollary 7.4]

Theorem A: Let $1 < p \le 2$. For every $\phi \in L^p_{\alpha,\beta}$ we have

$$\int_0^\infty x^{2(3\alpha+\beta)(p-2)}\left|h_{\alpha,\beta}(\phi)(x)\right|^p dv(x) \le C \int_0^\infty |\phi(x)|^p\, dv(x) \quad (1.1)$$

where C is a suitable positive constant depending only on p.

Our first aim in this paper is to give a sense to the inequality (1.1) when $0 < p \le 1$. Note that in general (1.1) is not true when $p = 1$. Indeed, define

$$\phi\,(x) = \begin{cases} 1, & x \in (0,1) \\ 0, & otherwise \end{cases}$$

Then according to $[11,\ p.22\ (6)]$, $h_{\alpha,\beta}(\phi)(y) = y^{-(3\alpha+\beta)}J_{3\alpha+\beta}(y),\ y \in (0,\infty)$. Moreover there exists $K > 0$ such that

$$\left|z^{-(3\alpha+\beta)}J_{3\alpha+\beta}(z)\right| \ge \frac{1}{2^{5\alpha+3\beta}\Gamma(5\alpha+3\beta)},\ \text{for every}\ z \in (0,k)$$

Thus, we have

$$\int_0^K \frac{dx}{x} \le 2^{5\alpha+3\beta}\Gamma(5\alpha+3\beta)\int_0^K\left|h_{\alpha,\beta}(\phi)(x)\right|\frac{dx}{x} \quad (1.2)$$

Suppose now that (1.1) holds for $p = 1$ and for every $\phi \in L^1_{\alpha,\beta}$. As $\phi \in L^1_{\alpha,\beta}$, we can write

$$\int_0^\infty\left|h_{\alpha,\beta}(\phi)(x)\right|\frac{dx}{x} \le C\int_0^1 dv(x) = \frac{C}{2^{3\alpha+\beta}\Gamma(5\alpha+3\beta)} (1.3)$$

for a certain $C > 0$. By combining (1.2) and (1.3) it concludes that

$$\int_0^K \frac{dx}{x} \le C.$$

Thus we get a contradiction

To study the inequality (1.1), when $0 < p \le 1$, inspired in celebrated and well-known results concerning to Fourier transforms ($[6]and\ [12,\ Chapter\ III]$), we need to introduce new

Hardy type function spaces. The Hankel translation [18] plays an important role in the definition of our atomic Hardy spaces. Haimo [16] and Hirschman [18] investigated a convolution operation and a translation operator associated to the Hankel transformation. If $f, g \in L^1_{\alpha,\beta}$, the Hankel type convolution $f\#g$ of f and g is defined by

$$(f\#g)(y) = \int_0^\infty f(x)\left(\tau_y\, g\right)(x)\, dv\,(x)\,, \qquad y \in (0,\infty)$$

where the Hankel type translation τ_y ; $y \in (0,\infty)$ is given by

$$(\tau_x g)(x) = \int_0^\infty D_{\alpha,\beta}(x,y,z)\, g(z)\, dv\,(z),\;\; x,y \in (0,\infty),$$

being

$$D_{\alpha,\beta}(x,y,z) = \frac{2^{\alpha-5\beta}\Gamma((3\alpha+\beta)^2)}{\Gamma(2\alpha)\sqrt{\pi}}(xyz)^{-2(\alpha-\beta)}A\,(x,y,z)^{-4\beta}, x,y,z \in (0,\infty),$$

and where $A\,(x,y,z)$ denotes the area of a triangle having sides with lengths x, y *and* z when such a triangle exists, and $A\,(x,y,z) = 0$, otherwise.

In [16] and [18] the Hankel convolution and Hankel translation were studied on the $L^p_{\alpha,\beta}$ −spaces. In recent years in [30] and [24], the # - convolution and the operator $\tau_y, y \in (0,\infty)$ have been studied in spaces of generalized functions with exponential and slow growth.

We now define our atomic Hardy type spaces.Firstly we introduce a class of fundamental functions that we will call atoms. Let $0 < p \leq 1$. A Lebesgue measurable function on $(0,\infty)$is a p-atom when a satisfies the following conditions

(i) there exists $a \in (0,\infty)$such that $b(x) = 0\,,\; x \geq a$.

(ii) $\|b\|_2 \leq \nu\,(0,a)^{\frac{1}{2}-\frac{1}{p}}$, where $a \in (0,\infty)$ is given in (i) ;

(iii) $\int_0^a x^{2j} b(x)\, dv(x) = 0$,for every $j = 0,1,\ldots.r$,

where $r = [(3\alpha+\beta)\,(1-p)/p]$. Here by $[x]$ we denote the integer part of x. By S_e we represent the function space that consists of all those even functions ϕ belonging to the Schwartz space S. S_e is endowed with the topology induced in it by S. As usual S'_e denotes the dual space of S_e.S'_e is equipped with the weak $*$ topology.

Let $0 < p \leq 1$. Our Hardy type space $\mathcal{H}_{p,\alpha,\beta}$ is constituted by all those $f \in S'_e$that can be represented by

$$f = \sum_{j=0}^{\infty} \lambda_j \tau_{x_j} a_j \quad (1.4)$$

being $x_j \in (0,\infty), \lambda_j \in \mathbb{C}$ and a_j is a p-atom, for every $j \in N$, where

$$\sum_{j=0}^{\infty} |\lambda_j|^p < \infty$$

and the series in (1.4) converges in S'_e .

We define on $\mathcal{H}_{p,\alpha,\beta}$ the quasinorm $\| \quad \|_{p,\alpha,\beta}$ by

$$\|f\|_{p,\alpha,\beta} = \inf\left(\sum_{j=0}^{\infty} |\lambda_j|^p\right)^{1/p} ,$$

where the infimum is taken over all those sequences $(\lambda_j)_{j=0}^{\infty} \subset \mathbb{C}$ such that f is given by (1.4) for certain $x_j \in (0,\infty)$and p-atoms a_j , $j \in N$.

By proceeding in a standard way (See [13], for instance)we can see that defining the metric $d_{p,\alpha,\beta}$on $\mathcal{H}_{p,\alpha,\beta}$ by

$$d_{p,\alpha,\beta}(f,g) = \|f-g\|_{p,\alpha,\beta}^p , \qquad f,g \in \mathcal{H}_{p,\alpha,\beta} ,$$

$\mathcal{H}_{p,\alpha,\beta}$ is complete, metric linear space. Moreover, $\mathcal{H}_{p,\alpha,\beta}$ is a quasi Banach space.In Section 2 we study the Hankel transformation on the Hardy type space $\mathcal{H}_{p,\alpha,\beta}$. In particular we establish the following extension of Theorem A to $0 < p \leq 1$.

Theorem 1.1: Let $0 < p \leq 1$. Then there exists $C > 0$ such that

$$\int_0^{\infty} |h_{\alpha,\beta}(f)(x)|^p x^{2(3\alpha+\beta)(p-2)}\, d\nu(x) \leq C\, \|f\|_{p,\alpha,\beta}^p ,$$

for every $f \in \mathcal{H}_{p,\alpha,\beta}$.

Note that the inequality showed in Theorem 1.1 can be seen as a Paley type inequality for Hankel transforms [12, p.55]. In [21] Y. Kanjin has obtained, for other variant of the Hankel transformation, an inequality similar to the one established in Theorem 1.1 that holds on classical Hardy spaces.

Following [4] we can prove the following result:

Theorem B ([4, Theorem 1.1]). Let $1 < p < \infty$. Assume that k is a locally integrable function on$(0,\infty)$and define the operator $T_k$by $T_k f = k \# f$. If the following two conditions

(i) there exists $C_p > 0$such that $\|T_k f\|_p \leq C_p \|f\|_p,\ f \in L^p_{\alpha,\beta}$,

(ii) there exist two positive constants A and B such that for every $x, y \in (0,\infty)$

$$\int \left|(\tau_x k)(z) - (\tau_y k)(z)\right| d\nu(z) \le A, |x-z| > B\,|y-x|$$

hold, then for every $1 < q < p$ there exists $C_q > 0$ for which

$$\|T_k f\|_q \le C_q \|f\|_q \ ,\ f \in L^q_{\alpha,\beta} \ ,$$

and there exists $C_1 > 0$ being

$$\gamma\left(\{x \in (0,\infty) : |T_k f(x)| > \lambda\}\right) \le \frac{C_1}{\lambda}\|f\|_1 \ , \qquad \lambda > 0 \ and \ f \in L^1_{\alpha,\beta} \ .$$

In Section 3 we study the Hankel type convolution operators on $\mathcal{H}_{p,\alpha,\beta}$.If $m \in L^\infty_{\alpha,\beta}$then m defines a Hankel type multiplier M_m through

$$M_m f = h_{\alpha,\beta}(m\, h_{\alpha,\beta}\, f).$$

In particular if $m \in L^1_{\alpha,\beta}$ and $h_{\alpha,\beta}(m) \in L^1_{\alpha,\beta}$, M_m coincides with the convolution operator $T_{h_{\alpha,\beta}}(m)$ ([18, Theorem 2d]). Gosselin and Stempak [14] obtained a Hankel version of the celebrated Mihlin-Hormander Fourier multiplier Theorem. The authors [4, Theorems 1.2 and 1.4] and Kapelko [20] have extended the multiplier theorem of Gosselin and Stempak in different ways. In Section 4, inspired in the ideas included in the papers of Coifman [7] and Miyachi [25], we study Hankel type multipliers in the space $\mathcal{H}_{1,\alpha,\beta}$.

2. The Hankel type transformation of $\mathcal{H}_{p,\alpha,\beta}$:

Our aim in this section is to study the Hankel type transformation on the Hardy type spaces $\mathcal{H}_{p,\alpha,\beta}$. Here we prove, as a main result, Theorem 1.1.Our results can be seen as a Hankel version of celebrated properties concerning Fourier transforms of classical Hardy spaces ([6], [8] and [12]).

Firstly we establish useful estimates for the Hankel type transform of p-atoms.

Lemma 2.1: Let $0 < p \le 1$. Then for every p-atom, we have

(i) $\left|h_{\alpha,\beta}(b)\right| \le C\, y^{2(r+1)}\|b\|_2^{-A}$, $y \in (0,\infty)$,

where $A = \{2(r+1)p + 2(3\alpha+\beta)(p-1)\} / \{(3\alpha+\beta)(2-p)\}$,

(ii) $\left|h_{\alpha,\beta}(b)(y)\right| \le C\, \|b\|_2^{2(p-1)/(p-2)}$, $y \in (0,\infty)$.

Proof: Let $b$be a p-atom. Assume that $a \in (0,\infty)$ is such that $b(x) = 0$, $x \ge a$ and

$$\|b\|_2 \le \gamma\left((0,a)\right)^{1/2-1/p} \quad (2.1)$$

(i) since

$$\int_0^\infty a(x)x^{2j}\, d\gamma(x) = 0,$$

for every $j \in N, 0 \le j \le r = [(3\alpha+\beta)(1-p)/p]$,

we can write

$$h_{\alpha,\beta}(b)(y) = \int_0^a (xy)^{-(\alpha-\beta)} J_{\alpha-\beta}(xy)\, b(x)x^{4\alpha}\, dx$$

$$= \int_0^a \left((xy)^{-(\alpha-\beta)} J_{\alpha-\beta}(xy) - \sum_{j=0}^r c_{j,\alpha,\beta}(xy)^{2j}\right) b(x)x^{4\alpha},$$

$y \in (0,\infty)$, where $c_{j,\alpha,\beta} = (-1)^j/\{2^{\alpha-\beta+2j}\Gamma(\alpha-\beta+j+1)j!\}$, $j = 0, \dots, r$.

Hence according to [22, (2.2)], from (1.5) it follows

$$\left|h_{\alpha,\beta}(b)(y)\right| \le C\, y^{2(r+1)} \int_0^a |b(x)| x^{2(r+1)} d\gamma(x)$$

$$\le C\, y^{2(r+1)} \|b\|_2 \left(\int_0^a x^{4(r+1)} d\gamma(x)\right)^{1/2}$$

$$\le C\, y^{2(r+1)} \|b\|_2 a^{2(r+1)+3\alpha+\beta} \le C\, y^{2(r+1)} \|b\|_2^{-A}, \qquad y \in (0,\infty),$$

being $A = \{2(r+1)p + 2(3\alpha+\beta)(p-1)\}/(3\alpha+\beta)(2-p)$,

(ii)By taking into account that the function $z^{-(\alpha-\beta)}J_{\alpha-\beta}(z)$is bounded on $(0,\infty)$, we can write

$$\left|h_{\alpha,\beta}(b)(y)\right| \le C \int_0^a |b(x)| x^{4\alpha} dx \le C\, \|b\|_2 a^{3\alpha+\beta}$$

$$\le C\, \|b\|_2^{2(p-1)/(p-2)}, \qquad y \in (0,\infty).$$

Thus proof is completed.

As a consequence of Lemma 2.1, we prove the following essential property.

Proposition 2.1: Let $0 < p \le 1$. If b is a p-atom then

$$\left|h_{\alpha,\beta}(\tau_x b)(y)\right| \le C\, y^{2(3\alpha+\beta)(1/p-1)}, x, y \in (0,\infty).$$

Proof : Let b a p-atom. Assume firstly that

$$y^{2(r+1)} \|b\|_2^{-A} \le \|a\|_2^{2(p-1)/(p-2)}$$, where $y \in (0,\infty)$ and as in Lemma 2.1

$A = \{2(r+1)p + 2(3\alpha+\beta)(p-1)\}/\{(3\alpha+\beta)(2-p)\}$. Then, from Lemma 2.1, (i) it infers that

$$\left|h_{\alpha,\beta}(b)(y)\right| \leq C\, y^{2(r+1)}\|b\|_2^{-A} \leq C\, y^{2(3\alpha+\beta)(1/p-1)}, \qquad y \in (0,\infty).$$

On the other hand, if $y^{2(r+1)}\|b\|_2^{-A} \geq \|b\|_2^{2(p-1)/(p-2)}$ then Lemma 2.1, (ii), leads to

$$\left|h_{\alpha,\beta}(b)(y)\right| \leq C\, \|b\|_2^{2(p-1)/(p-2)} \leq C\, y^{2(3a+\beta)(1/p-1)}, \qquad y \in (0,\infty).$$

Thus we have proved that

$$\left|h_{\alpha,\beta}(b)(y)\right| \leq C\, y^{2(3a+\beta)(1/p-1)},\ y \in (0,\infty) \quad (2.2)$$

According to [24, (2.1)]

$$h_{\alpha,\beta}(\tau_x b)(y) = 2^{\alpha-\beta}\Gamma(3\alpha+\beta)(xy)^{-(\alpha-\beta)}J_{\alpha-\beta}(xy)h_{\alpha,\beta}(b)(y),\ x,y \in (0,\infty). \quad (2.3)$$

Note that here C is a positive constant that is not depending on $x, y \in (0,\infty)$. Thus proof of proposition is completed.

The Hankel type transformation $h_{\alpha,\beta}$ is an automorphism of $S_e([1, Satz\ 5]\ and$ [10, *p.81]*. The transformation $h\alpha,\beta$ is defined on the dual space Se' by transposition. That is, if $f \in S_e'$, $h_{\alpha,\beta}\ f$ is the element of S_e' defined by

$$\langle h_{\alpha,\beta}\ f, \phi\rangle = \langle f,\ h_{\alpha,\beta}\ \phi\rangle, \qquad \phi \in S_e.$$

Thus as it is well-known, $h_{\alpha,\beta}$ is an automorphism of S_e'. Hence if $f \in \mathcal{H}_{p,\alpha,\beta}$, with $0 < p < 1$ and f admits the representation (1.4) where $x_j \in (0,\infty)$, $\lambda_j \in \mathbb{C}$ and b_j is a p-atom, for every $j \in \mathbb{N}$, and

$$\sum_{j=0}^{\infty}\left|\lambda_j\right|^p < \infty,$$

then, according to (2.3),

$$h_{\alpha,\beta}(f)(y) = 2^{\alpha-\beta}\Gamma(3\alpha+\beta)\sum_{j=0}^{\infty}\lambda_j\left(x_j y\right)^{-(\alpha-\beta)}J_{\alpha-\beta}\left(x_j y\right)h_{\alpha,\beta}\left(b_j\right)(y), \quad (2.4)$$

$$y \in (0,\infty).$$

Moreover, since

$$\sum_{j=0}^{\infty}\left|\lambda_j\right| \leq \left(\sum_{j=0}^{\infty}\left|\lambda_j\right|^p\right)^{1/p},$$

From Proposition 2.1, it deduces that $h_{\alpha,\beta}\ f$ is a continuous function on $(0,\infty)$ and that

$$\left|h_{\alpha,\beta}(f)(y)\right| \leq C\left(\sum_{j=0}^{\infty}\left|\lambda_j\right|^p\right)^{1/p} y^{2(3\alpha+\beta)(1/p-1)}, y \in (0,\infty).$$

Hence we can conclude that

$$y^{-2(3\alpha+\beta)(1/p-1)}\left|h_{\alpha,\beta}(f)(y)\right| \leq C\,\|f\|_{p,\alpha,\beta}\,,\ y \in (0,\infty). \quad (2.5)$$

From (2.5) it informs the following weak type inequality for the Hankel type transformation $h_{\alpha,\beta}$.

Proposition 2.2: Let $0 < p \leq 1$. There exists $C > 0$ such that for every $f \in \mathcal{H}_{p,\alpha,\beta}$

$$\gamma(\{y \in (0,\infty) : \left|h_{\alpha,\beta}(f)(y)\right| y^{2(3\alpha+\beta)(1-2/p)} > \lambda\}) \leq C\,\frac{\|f\|_{p,\alpha,\beta}^p}{\lambda^p}, \lambda \in (0,\infty).$$

Proof: Let $f \in \mathcal{H}_{p,\alpha,\beta}$ and $\lambda \in (0,\infty)$ .By (2.5) it follows

$$\gamma\left(\{y \in (0,\infty): \left|h_{\alpha,\beta}(f)(y)\right| y^{2(3\alpha+\beta)(1-2/p)} > \lambda\}\right) \leq \int_0^{(C\,\|f\|_{p,\alpha,\beta}/p)^{p/(6\alpha+2\beta)}} d\gamma(y)$$

$$\leq C\,\frac{\|f\|_{p,\alpha,\beta}^p}{\lambda^p}.$$

Thus proof is completed.

To establish Theorem 1.1, next lemma is fundamental.

Lemma 2.2: Let $0 < p \leq 1$. There exists $C > 0$such that, for every p-atom,

$$\int_0^{\infty}\left|h_{\alpha,\beta}(b)(y)\right|^p y^{2(3\alpha+\beta)(p-2)} d\gamma\,(y) \leq C\,.$$

Proof: Let b be a p-atom. Assume that $R > 0$. By virtue of Lemma 2.1, (i), since

$$r > \{(3\alpha+\beta)(1-p)/p\} - 1,$$

we can write

$$\int_0^R \left|h_{\alpha,\beta}(b)(y)\right|^p y^{2(3\alpha+\beta)(p-2)}\, d\gamma\,(y) \leq C \int_0^R y^{2(r+1)p+2(3\alpha+\beta)(p-2)} d\gamma\,(y) \|b\|_2^{-Ap}$$

$$\leq C\left(R\,\|b\|_2^{p/[(3\alpha+\beta)(p-2)]}\right)^{2[(r+1)p+(3\alpha+\beta)(p-1)]} \quad (2.6)$$

Also according to [17, Theorem 3], Holder's inequality leads to

$$\int_R^{\infty}\left|h_{\alpha,\beta}(b)(y)\right|^p y^{2(3\alpha+\beta)(p-2)}\, d\gamma\,(y)$$

$$\leq \left\{ \int_0^\infty \left| h_{\alpha,\beta}(b)(y) \right|^2 d\gamma(y) \right\}^{p/2} \left\{ \int_R^\infty y^{-4(3\alpha+\beta)} d\gamma(y) \right\}^{(2-p)/2}$$

$$\leq C \, \|b\|_2^p R^{-(3\alpha+\beta)(2-p)}. \quad (2.7)$$

Now taking $R = \|b\|_2^{p/[(3\alpha+\beta)(2-p)]}$, from (2.6) and (2.7), we conclude that

$$\int_0^\infty \left| h_{\alpha,\beta}(b)(y) \right|^p y^{2(3\alpha+\beta)(p-2)} d\gamma(y) \leq C$$

Thus proof is completed.

Now we prove Theorem 1.1

Proof of Theorem 1.1: Let $0 < p < 1$ *and* $f \in \mathcal{H}_{p,\alpha,\beta}$. Assume that f is given by (1.4). Then $h_{\alpha,\beta}(f)$admits the representation (2.4) for certain $x_j \in (0,\infty)$, $\lambda_j \in \mathbb{C}$ *and* a_j p-atom, for each $\in \mathbb{N}$, and being

$$\sum_{j=0}^{\infty} |\lambda_j|^p < \infty .$$

According to Lemma 2.2 and since the function $z^{-(\alpha-\beta)} J_{\alpha-\beta}(z)$ is bounded on $(0,\infty)$ we can write

$$\int_0^\infty \left| h_{\alpha,\beta}(b)(y) \right|^p y^{2(3\alpha+\beta)(p-2)} d\gamma(y) \leq C \sum_{j=0}^{\infty} |\lambda_j|^p \int_0^\infty \left| h_{\alpha,\beta}(a_j)(y) \right|^p \times y^{2(3\alpha+\beta)(p-2)} d\gamma(y)$$

$$\leq C \sum_{j=0}^{\infty} |\lambda_j|^p .$$

Thus

$$\int_0^\infty \left| h_{\alpha,\beta}(f)(y) \right|^p y^{2(3\alpha+\beta)(p-2)} d\gamma(y) \leq C \, \|f\|_{p,\alpha,\beta}^p .$$

This completes the proof of theorem.

A Hankel version of the Hardy inequality appears when we take $p = 1$ in Theorem 1.1

Corollary 2.1: There exists $C > 0$ such that

$$\int_0^\infty \left| h_{\alpha,\beta}(f)(y) \right| \frac{dy}{y} \leq C \, \|f\|_{1,\alpha,\beta} ,$$

for every $f \in \mathcal{H}_{1,\alpha,\beta}$.

Finally, from a Paley-Wiener type theorem for the Hankel transform due to Griffith [15], we can deduce a characterization of the distributions in $\mathcal{H}_{p,\alpha,\beta}$through Hankel transforms.

Let b be a p-atom. Assume that $a \in (0,\infty)$is such that $b(x) = 0$, $x \geq a$, and $\|b\|_2\, \gamma\left((0,a)\right)^{\frac{1}{2}-\frac{1}{p}}$. Then according to [17, Theorem 3], it follows

$$\left\|h_{\alpha,\beta}(b)\right\|_2 = \|b\|_2 \leq \gamma\big((0,a)\big)^{\frac{1}{2}-\frac{1}{p}}.$$

Moreover by taking into account well-known properties of the Bessel functions [30,5.1 (6) and (7)] we can write

$$\Delta^j_{\alpha,\beta} h_{\alpha,\beta}(b) = 0\,, \qquad j = 0, \ldots\ldots, r,$$

where $\Delta_{\alpha,\beta} = x^{4\beta-2} D_x x^{4\alpha} D_x$ and $r = [(3\alpha+\beta)(1-p)/p]$, where $D_x = \frac{d}{dx}$.

Also by [15], $h_{\alpha,\beta}(b)$ is an even and entire function such that

$$\left|h_{\alpha,\beta}(b)(z)\right| \leq C\, e^{a(Imz)}, \qquad Z \in \mathbb{C}\,.$$

To simplify we will say that an even and entire function A is p −normalized and of exponential type $a \in (0,\infty)$ when A satisfies the following conditions.

(i) $\|A\|_2 \leq \gamma\,(0,a)^{\frac{1}{2}-\frac{1}{p}}$,

(ii) $\Delta^j_{\alpha,\beta}\, A(0) = 0$, $j = 0,1, \ldots., r$ being $\Delta_{\alpha,\beta}$ and r as above, and

(iii) $|A(z)| = O\left(e^{a|Imz|}\right)$, as $|z| \to \infty$.

In other words, we have proved that if b is a p-atom,$h_{\alpha,\beta}(b)$ is p-normalized and of exponential type a, for some $a \in (0,\infty)$.

Conversely, suppose that an even and entire function A is p-normalized and of exponential type $a \in (0,\infty)$. Then Griffith's Theorem [15] implies that $h_{\alpha,\beta}(A)(x) = 0,\ x \geq a$, and that

$$\left\|h_{\alpha,\beta}(A)\right\|_2 \leq \gamma\left((0,a)\right)^{\frac{1}{2}-\frac{1}{p}}.$$

Moreover,

$$h_{\alpha,\beta}\left(h_{\alpha,\beta}(A)\right) = A \text{ and } \Delta^j_{\alpha,\beta}\, A\,(0) = (-1)^j \int_0^a x^{2j}\, h_{\alpha,\beta}(A)(x)\, d\gamma(x) = 0, \qquad j = 0, \ldots\ldots, r.$$

Thus by taking into account (2.3) we can conclude the following characterization of the distributions in $\mathcal{H}_{p,\alpha,\beta}$.

Proposition 2.3: Let $0 < p < 1$. A distribution $f \in S'_e$is in $\mathcal{H}_{p,\alpha,\beta}$ if and only if, there exist $x_j \in (0,\infty)$, $\lambda_j \in \mathbb{C}$ and a p-normalized and of exponential type a_j function $A_j, a_j \in (0,\infty)$, for every $j \in \mathbb{N}$, such that

$$h_{\alpha,\beta}(f)(y) = \sum_{j=0}^{\infty} \lambda_j (xjy)^{-(\alpha-\beta)} J_{\alpha-\beta}(xjy) A_j(y), \qquad y \in (0,\infty),$$

and that

$$\sum_{j=0}^{\infty} |\lambda_j|^p < \infty .$$

3. Hankel type convolution operators in the spaces $\mathcal{H}_{p,\alpha,\beta}$:

In this section we study Hankel type convolution operators defined by

$$T_k f = k \# f,$$

where k is a locally integrable function on $(0,\infty)$, on the hardy type spaces $\mathcal{H}_{p,\alpha,\beta}$. According to [1], and [10] the topology of S_e is also generated by the family $\{\rho_{m,n}\}_{m,n\in\mathbb{N}}$of seminorms, where

$$\rho_{m,n}(\phi) = \underset{x \in (\infty)}{Sup} \left| x^m \Delta_{\alpha,\beta}^n \phi(x) \right|, \ \phi \in S_e, \ m,n \in \mathbb{N},$$

where $\Delta_{\alpha,\beta} = x^{4\beta-2} D_x x^{4\alpha} D_x$, then $\{\eta_{m,n}^{\alpha,\beta}\}_{m,n\in\mathbb{N}}$generates the topology of S_e . Hence from [24, Proposition 4.2] we can deduce characterization of the Hankel type convolution operators on $S_e \; and \; S'_e$.

Our first result is an extension of Theorem B.

Proposition 3.1: Let k be a locally integrable function on $(0,\infty)$. Assume that the following two conditions

(i) T_k defines a bounded linear operator from $L^2_{\alpha,\beta}$ into itself.

(ii) There exist two positive constants A and B such that

$$\int_{|x-z|>B\,|y-x|} \left|(\tau_x k)(z) - (\tau_y k)(z)\right| \, d\gamma(z) \le A, \qquad x,y \in (0,\infty),$$

and, for a certain $c > 1$,

$$\int_{cR}^{\infty} |(\tau_x k)(z) - k(z)| \, d\gamma(z) \le A, \ \ x \in (0,R) \; and \; R \in (0,\infty),$$

hold. Then T_k defines a bounded linear mapping from $\mathcal{H}_{1,\alpha,\beta}$ into $L^1_{\alpha,\beta}$.

Proof: Let $b$be a 1-atom. We choose $a > 0$ such that

$$b(x) = 0,\ x \geq a,\ and\ \|b\|_2 \leq \gamma\left((0,a)\right)^{-\frac{1}{2}}.$$

We can write

$$\int_0^\infty |(T_k b)(x)|\, d\gamma(x) = \left(\int_0^{ca} + \int_{ca}^\infty\right)|(T_k b)(x)|\, d\gamma(x) = I_1 + I_2\,.$$

Here $c > 1$ is the one given in (ii).

Since T_k is a bounded operator from $L^2_{\alpha,\beta}$ into itself, Holder's inequality leads to

$$\int_0^{ca} |(T_k b)(x)|\, d\gamma(x) \leq \left\{\int_0^\infty |(T_k b)(x)|^2\, d\gamma(x)\right\}^{\frac{1}{2}} \left\{\int_0^{ca} d\gamma(x)\right\}^{\frac{1}{2}}$$
$$\leq C\,\|b\|_2 a^{3\alpha+\beta} \leq C$$

Also by taking into account that $\int_0^\infty b(y)\, d\gamma(y) = 0$,the condition (ii) allows us to write

$$\int_{Ca}^\infty |(T_k b)(x)|\, d\gamma(x) = \int_{Ca}^\infty \left|\int_0^\infty (T_k b)(y)\, b(y) d\gamma(y)\right| d\gamma(x)$$
$$= \int_{Ca}^\infty \left|\int_0^\infty [(\tau_x k)(y) - k(x)]\, b(y)\, d\gamma(y)\right| d\gamma(x)$$
$$\leq \int_0^a |b(y)| \int_{ca}^\infty \left|(\tau_y k)(x) - k(x)\right| d\gamma(x) d\gamma(y) \leq C \int_0^a |b(y)|\, d\gamma(y)$$
$$\leq C\,\|b\|_2 \left\{\int_0^a d\gamma(y)\right\}^{\frac{1}{2}} \leq C.$$

Hence it concludes that

$$\|T_k b\| \leq C.$$

Note that the positive constant C not depending on the 1-atom b. Moreover, according to (2.3) [18, Theorem 2d] and [29,p.16],

$$\|T_k(\tau_x b)\|_1 = \|k\#\tau_x b\|_1 = \|\tau_x(k\#b)\|_1 \leq \|k\#b\|_1 \leq C, \quad (3.1)$$

for every $x \in (0,\infty)$.

Let now f be in $\mathcal{H}_{1,\alpha,\beta}$. Then $f \in S'_e$ and

$$f = \sum_{j=0}^\infty \lambda_j \tau_{xj} b_j\ ,(3.2)$$

where $\lambda_j \in \mathbb{C}$, $x_j \in (0,\infty)$ and b_j is a 1-atom, for every $j \in \mathbb{N}$, and

$$\sum_{j=0}^{\infty} |\lambda_j| < \infty.$$

Series in (3.2) converges in $L^1_{\alpha,\beta}$. Indeed, it is sufficient to note that, according to again [29, p.16]

$$\|\tau_x b\| \le \|b\|_1 \le 1,$$

for every $x \in (0,\infty)$and every 1-atom b. Hence $f \in L^1_{\alpha,\beta}$.

By virtue of Theorem B, $\tau_x f$ is in weak$L^1_{\alpha,\beta}$and

$$\tau_x f = \sum_{j=0}^{\infty} \lambda_j T_k \tau_{x_j} b_j \text{ .(3.3)}$$

By (3.1) the series in (3.3) converges in $L^1_{\alpha,\beta}$ and

$$\|T_k f\|_1 \le C \sum_{j=0}^{\infty} |\lambda_j| .$$

Hence

$$\|T_k f\|_1 \le C \|f\|_{1,\alpha,\beta} .$$

Thus proof is completed.

The following result can be established by proceeding as in the proof of Proposition 3.1.

Proposition 3.2: Let k be a locally integrable function on $(0,\infty)$. Assume that the following three conditions are satisfied.

(i) $T_k$defines a bounded linear operator from $L^2_{\alpha,\beta}$into itself

(ii) T_k defines a bounded linear operator from $L^1_{\alpha,\beta}$ into S'_e

(iii) There exist $A > 0$ and $c > 1$ such that

$$\int_{CR}^{\infty} |(\tau_x k)(z) - k(z)| d\gamma(z) \le A, \quad x \in (0,R) \; and \; R \in (0,\infty).$$

Then T_k is a bounded linear mapping from $\mathcal{H}_{1,\alpha,\beta}$ into $L^1_{\alpha,\beta}$.

Proof: It is enough to proceed as in the proof of Proposition 3.1. Here the condition (ii) replaces to (1,1) weak type for the operator T_k that it is used in the proof of Proposition 3.1. Thus proof is completed.

We now describe some sets of functions that define Hankel type convolution operators between Hardy type spaces $\mathcal{H}_{p,\alpha,\beta}$.The corresponding results for the usual convolution operator on classical Hardy spaces were established by Colzani [9].

Proposition 3.3: Let $0 < p \leq q \leq 1$. Assume that, for every $n \in \mathbb{N}$, $x_n, \mathcal{E}_n \in (0,\infty)$, and g_n is a function that satisfies the following properties.

(i) $g_n(x) = 0,\ x \geq 2^{-n}$;

(ii) $\|g_n\|_1 \leq \epsilon_n 2^{2(3\alpha+\beta)(1/q-1/p)n}$, and

(iii) $\left\|t^{2(3\alpha+)(1/p-1)} h_{\alpha,\beta}(g_n)\right\|_2 \leq \epsilon_n 2^{2(3\alpha+\beta)(1/q-1/2)n}$.

Suppose also that there exists $C > 0$ such that $x_n \leq C\, 2^{-n}, n \in \mathbb{N}$, and

$$\sum_{n=0}^{\infty} \epsilon_n^q < \infty$$

and define

$$k = \sum_{n=0}^{\infty} \tau_{x_n} g_n.$$

Then T_k defines a bounded linear mapping from $\mathcal{H}_{p,\alpha,\beta}$ into $\mathcal{H}_{q,\alpha,\beta}$.

Proof: Firstly note that according to [18, Theorem 2b and Theorem 2d] and by (2.3), we can write

$$T_k b = \sum_{n=0}^{\infty} \tau_{x_n}(b\# g_n).$$

Let $n \in \mathbb{N}$.

Suppose that $b(x) = 0,\ x \geq a$ and that $\|b\|_2 \leq \gamma\left((0,a)\right)^{\frac{1}{2}-\frac{1}{p}}$, where $a > 0$. Then $\left(\tau_{x_n}(b\# g_n)\right)(x) = 0,\ x \geq a + 2^{-n} + x_n$. Infact, we have

$$(\tau_y g_n)(z) = \int_{|y-z|}^{y+z} D_{\alpha,\beta}(y,z,u) g_n(u)\, d\gamma(u) = 0,\ \ |y-z| \geq 2^{-n}.$$

Hence,

$$\|(b\# g_n)(y) = \int_0^a b(z)\left(\tau_y g_n\right)(z)\, d\gamma(z) = 0, \qquad y \geq a + 2^{-n},$$

and then

$$\left(\tau_{x_n}(b\# g_n)\right)(x) = \int_{|x_n - x|}^{x_n + x} D_{\alpha,\beta}(x_n, x, y)(b\# g_n)(y)\, d\gamma(y) = 0,$$

$x \geq a + 2^{-n} + x_n$.

Moreover, since

$$\int_0^\infty x^{2j}\, b(x) d\gamma(x) = 0\,, \quad j = 0, \dots\dots, r\,, \text{ being}$$

$r = [(3\alpha+\beta)(1-p)/p]$, we have that

$$\int_0^\infty x^{2j} (b\#g_n)(x)\, d\gamma(x) = 0, \quad j = 0, \dots\dots, r.$$

Indeed, let $j = 0, \dots\dots, r$. Fubini's Theorem leads to

$$\int_0^\infty x^{2j} (b\#g_n)\,(x)\, d\gamma(x)$$

$$= \int_0^\infty \int_0^\infty \int_0^\infty x^{2j}\, b(y) g_n(z) D_{\alpha,\beta}\,(x,y,z)\, d\gamma\,(z)\, d\gamma(y)\, d\gamma(x)$$

$$= \int_0^\infty b(y) \int_0^\infty g_n(z) \int_0^\infty x^{2j} D_{\alpha,\beta}(x,y,z)\, d\gamma(x)\, d\gamma(z)\, d\gamma(y). \quad (3.4)$$

We now evaluate the integral

$$\int_0^\infty x^{2j} D_{\alpha,\beta}(x,y,z)\, d\gamma(x), \qquad y,z \in (0,\infty).$$

Let $y,z \in (0,\infty)$. We can write, for certain $b_{i,j} \in \mathbb{R}$, $i = 0, \dots\dots, j$,

$$\int_0^\infty x^{2j} D_{\alpha,\beta}(x,y,z)\, d\gamma(x)$$

$$= \lim_{t\to 0+} 2^{\alpha-\beta}\Gamma(3\alpha+\beta) \int_0^\infty x^{2j} (xt)^{-(\alpha-\beta)} J_{\alpha-\beta}(xt) D_{\alpha,\beta}(x,y,z)\, d\gamma(x)$$

$$= \lim_{t\to 0+} (-1)^j 2^{\alpha-\beta}\Gamma(3\alpha+\beta)\Delta^j_{\alpha,\beta,t} \int_0^\infty (xt)^{-(\alpha-\beta)} J_{\alpha-\beta}(xt) D_{\alpha,\beta}(x,y,z)\, d\gamma(x)$$

$$= \lim_{t\to 0+} (-1)^j 2^{2(\alpha-\beta)}\Gamma(3\alpha+\beta)^2 \Delta^j_{\alpha,\beta,t}\, [(yt)^{-(\alpha-\beta)} J_{\alpha-\beta}(yt)(zt)^{-(\alpha-\beta)} J_{\alpha-\beta}(zt)]$$

$$= (-1)^j 2^{2(\alpha-\beta)}\Gamma(3\alpha+\beta)^2 \lim_{t\to 0+} \sum_{i=0}^{j} b_{i,j} t^{2i} \left(\frac{1}{t}\frac{d}{dt}\right)^{i+j} \begin{bmatrix} (yt)^{-(\alpha-\beta)} J_{\alpha-\beta}(yt) \\ \times\ (zt)^{-(\alpha-\beta)} J_{\alpha-\beta}(zt) \end{bmatrix}$$

$$= (-1)^j 2^{2(\alpha-\beta)}\Gamma(3\alpha+\beta)^2 \lim_{t\to 0+} \sum_{i=0}^{j} b_{i,j} t^{2j} \sum_{l=0}^{i+j} \binom{i+j}{l} (yt)^{-\alpha+\beta-l} J_{\alpha-\beta+l}(yt)$$

$$\times\ (-y^2)^{\ell} (zt)^{-(\alpha-\beta)-(i+j-l)} J_{\alpha-\beta+i+j-l}(zt) (-z^2)^{i+j-l}$$

$$= \Gamma(3\alpha+\beta)^2 \mathrm{b}_0, \mathrm{j} \sum_{l=0}^{j} \binom{j}{l} \frac{y^{2l}}{2^j \Gamma(\alpha-\beta+l+1)\Gamma(\alpha-\beta+j-l+1)}.$$

Hence by (3.4)

$$\int_0^\infty x^{2j}(b\#g_n)(x)\,d\gamma(x) = \frac{\Gamma(3\alpha+\beta)^2 b_0, j}{2^j}$$

$$\times \sum_{l=0}^{j} \binom{j}{l} \frac{1}{\Gamma(\alpha-\beta+l+1)\Gamma(\alpha-\beta+j-l+1)}$$

$$\times \int_0^\infty b(y) y^{2l} d\gamma(y) \int_0^\infty g_n(z) z^{2(j-l)} d\gamma(z) = 0.$$

By proceeding in a similar way to above we obtain

$$\int_0^\infty x^{2j}\left(\tau_{x_n}(b\#g_n)\right)(x)\,d\gamma(x) = \frac{\Gamma(3\alpha+\beta)^2 b_o, j}{2^j}$$

$$\times \sum_{l=0}^{j} \binom{j}{l} \frac{x_n^{2(j-l)}}{\Gamma(\alpha-\beta+l+1)\,\Gamma(\alpha-\beta+j-\ell+1)}$$

$$\times \int_0^\infty y^{2l}(b\#g_n)(y)\,d\gamma\,(y) = 0\,.$$

We conclude that, for some $p_n > 0, \tau_{x_n}(b\#g_n)/p_n$is a q-atom.

We shall now determinate p_n .

Firstly let us consider that $a \geq 2^{-n}$. According to [18, Theorem 2b], it follows

$$\|b\#g_n\|_2 \leq \|b\|_2 \|g_n\|_1 \leq \gamma\big((0,a)\big)^{1/2-1/p} \varepsilon_n 2^{-2n(3\alpha+\beta)(1/p-1/q)}$$

$$\leq C\,\varepsilon_n\,\gamma\left((0,a+2^{-n})\right)^{1/2-1/q}.$$

Here C is not depending on n or b .

Assume now that $a < 2^{-n}$. By taking into account that

$$\int_0^\infty y^{2j}\,b(y)\,d\gamma(y) = 0,\ \ j = 0, \ldots\ldots, r\,,\ \ being\ \ r = [(3\alpha+\beta)\,(1-p)/p],$$

we have

$$b\,\#\,g_n(x) = \int_0^\infty b(y)\left[(\tau_x g_n)(y) - \sum_{l=0}^{\gamma} \frac{\Gamma(3\alpha+\beta)\big(\Delta_{\alpha,\beta}^l g_n\big)(x) y^{2l}}{2^{2l}\, l!\ \Gamma(l+3\alpha+\beta)}\right] d\gamma(y)$$

$x \in (0,\infty)$.

Thus, since $h_{\alpha,\beta}$ is an isometry on $L^2_{\alpha,\beta}$ and by taking into account (2.3), it infers

$$\|b\#g_n\|_2 \le \int_0^\infty |b(y)| \left\| \tau_y g_n - \sum_{l=0}^{r} \frac{y^{2l}\Gamma(3\alpha+\beta)}{2^{\alpha-\beta} l!\,\Gamma(\alpha-\beta+l+1)} \Delta^l_{\alpha,\beta} g_n \right\|_2 d\gamma(y)$$

$$= \int_0^\infty |b(y)| \left\| \left(2^{\alpha-\beta}\Gamma(3\alpha+\beta)(xy)^{-(\alpha-\beta)} J_{\alpha-\beta}(xy) - \sum_{l=0}^{r} \frac{(-1)^l \Gamma(3\alpha+\beta)(xy)^{2l}}{2^{2l} l!\ \Gamma(\alpha-\beta+l+1)} \right) \times h_{\alpha,\beta}(g_n) \right\|_2 d\gamma(y)\,.$$

Moreover by [22, (2.2)] it follows

$$\|b\#g_n\|_2 \le C \int_0^\infty |b(y)|\, y^{2(3\alpha+\beta)(1/p-1)} \left\| x^{2(3\alpha+\beta)(1/p-1)} h_{\alpha,\beta}(g_n)(x) \right\|_2 d\gamma\,(y)$$

$$\le C \int_0^a |b(y)| y^{2(3\alpha+\beta)(1/p-1)} d\gamma\,(y) \varepsilon_n 2^{2(3\alpha+\beta)(1/q-1/2)n}$$

$$\le C\, \|b\|_2 \left\{ \int_0^a y^{4(3\alpha+\beta)(1/p-1)} d\gamma(y) \right\}^{1/2} \varepsilon_n 2^{2(3\alpha+\beta)(1/q-1/2)n}$$

$$\le C\, a^{2(3\alpha+\beta)(1/2-1/p)} a^{2(3\alpha+\beta)(1/p-1)+(3\alpha+\beta)} \varepsilon_n 2^{2(3\alpha+\beta)(1/q-1/2)n}$$

$$= C\, \varepsilon_n 2^{2(3\alpha+\beta)(1/q-1/2)n} \le C\, \varepsilon_n\, \gamma\left((0,a+2^{-n})\right)^{1/2-1/q},$$

where again $C$is not depending on $n\ or\ b$.

Now, since there exists $C>0$ such that $x_n \le C\,2^{-n},\ for\ every\ n \in \mathbb{N},\ by\ [29, p.16],$ it has

$$\left\| \tau_{x_n}(b\#g_n) \right\|_2 \le \|b\#g_n\|_2 \le C\, \varepsilon_n\, \gamma\,(0, a+2^{-n}+x_n)^{1/2-1/q}.$$

Then $p_n = C\,\varepsilon_n$ where C does not depend on $n\ or\ b$.

Thus we conclude that $T_k b \in \mathcal{H}_{\alpha,\beta,q}$ and

$$\|T_k b\|_{q,\alpha,\beta} \le C \left\{ \sum_{n=0}^{\infty} \varepsilon_n^q \right\}^{1/q}.$$

Let now $f \in \mathcal{H}_{\alpha,\beta,p}$, being

$$f = \sum_{j=0}^{\infty} \lambda_j \tau_{y_j} a_j \ ,$$

where $y_j \in (0, \infty)$, $\lambda_j \in \mathbb{C}$ and a_j is a p-atom, for every $j \in \mathbb{N}$, and such that

$$\sum_{j=0}^{\infty} |\lambda_j|^p < \infty \, .$$

Since the last series converges in $L^1_{\alpha,\beta}$ and $k \in L^1_{\alpha,\beta}$, by taking into account [18, Theorem 2b]

$$T_k f = \sum_{j=0}^{\infty} \lambda_j \tau_{y_j} T_k a_j \, .$$

Then we obtain that

$$\|T_k f\|_{q,\alpha,\beta} \leq C \left(\sum_{n=0}^{\infty} |\varepsilon_n|^q \right)^{1/q} \|f\|_{p,\alpha,\beta} \, .$$

Thus proof is completed.

4. Hankel type multipliers on Hardy type spaces $\mathcal{H}_{1,\alpha,\beta}$:

In this section we study Hankel type multipliers on Hardy type spaces $\mathcal{H}_{1,\alpha,\beta}$. Let m be a measurable bounded function on (0,∞). According to [17, Theorem 3] the operator M_m defined by

$$M_m f = h_{\alpha,\beta} \left(m \, h_{\alpha,\beta}(f) \right)$$

is linear and bounded from $L^2_{\alpha,\beta}$ into itself. In [4], [14] and [20] Hankel versions of Mihlin-Hormander multiplier theorem have been obtained. Here we establish a Mihlin-Hormander theorem for Hankel multipliers in a certain subspace of $\mathcal{H}_{1,\alpha,\beta}$. Note firstly that, according to (2.5), if $f \in \mathcal{H}_{p,\alpha,\beta}$, $0 < p \leq 1$, then $M_m f$ is in $S'_e$and it is defined by

$$\langle M_m f, \phi \rangle = \int_0^{\infty} m(y) h_{\alpha,\beta}(f)(y)(\phi)(y) \; d\gamma \,(y), \qquad \phi \in S_e$$

Moreover, we have

$$|\langle M_m f, \phi \rangle| \leq C \, \|f\|_{p,\alpha.\beta} \int_0^{\infty} y^{2(3\alpha+\beta)(1/p-1)} \left| h_{\alpha,\beta}(\phi)(y) \right| d\gamma(y), \qquad \phi \in S_e \, .$$

Hence M_m is a bounded operator from $\mathcal{H}_{p,\alpha,\beta}$ into S'_e.

To establish our Hankel multiplier theorem that it is inspired in the results about Fourier multipliers due to Miyachi [25], we need to introduce a subspace of $\mathcal{H}_{1,\alpha,\beta}$.

We say that a measurable function b on $(0,\infty)$ is a $(1,\infty)-$ atom when b is a 1-atom and $\|b\|_\infty \le \gamma\left((0,a)\right)^{-1}$, where $a \in (0,\infty)$is such that $\phi(x)=0,\ x \ge a$. Note that if $\|b\|_\infty \le \gamma\left((0,a)\right)^{-1}$ *and* $\phi(x)=0$, $x \ge a$, where $a \in (0,\infty)$, then

$$\|b\|_2 \le \|b\|_\infty \gamma\left((0,a)\right)^{1/2} \le \gamma\left((0,a)\right)^{-1/2}.$$

The space $\mathcal{H}_{1,\alpha,\beta}$ consists of all those $f \in L^1_{\alpha,\beta}$ being

$$f = \sum_{j=0}^{\infty} \lambda_j \tau_{x_j} b_j \qquad (4.1)$$

where the series converges in S'_e *and* $\lambda_j \in \mathbb{C}$, $x_j \in (0,\infty)$ and b_j is a $(1,\infty)-$atom for every $j \in \mathbb{N}$, and being

$$\sum_{j=0}^{\infty} |\lambda_j| < \infty.$$

Note that the series in (4.1) also converges in $L^1_{\alpha,\beta}$.

We define on $\mathcal{H}^{\infty}_{1,\alpha,\beta}$ the topology induced by the quasi norm $\|.\|^{\infty}_{1,\alpha,\beta}$ defined by

$$\|f\|^{\infty}_{1,\alpha,\beta} = \inf\left\{\sum_{j=0}^{\infty} |\lambda_j|\right\},\ f \in \mathcal{H}^{\infty}_{1,\alpha,\beta},$$

where infimum is taken over all those absolutely convergent complex sequences $(\lambda_j)_{j=1}^{\infty}$ for which the representation (4.1) holds for some $x_j \in (0,\infty)$ *and* $(1,\infty)-$atoms b_j , $j \in \mathbb{N}$.

It is not hard to see that $\mathcal{H}^{\infty}_{1,\alpha,\beta}$ is contained in $\mathcal{H}_{1,\alpha,\beta}$ and the topology of $\mathcal{H}^{\infty}_{1,\alpha,\beta}$is weaker than the one induced in it by $\mathcal{H}_{1,\alpha,\beta}$.

We are now in a position to establish our Hankel version of Mihlin-Hormander theorem on Hardy type spaces.

Theorem 4.1: Assume that $b \ge 0,\ s \ge 0,\ k \in \mathbb{N},\ k > (3\alpha+\beta)/2$ and $0 < s - b(2k + 3\alpha +$ β<2 .Suppose also that $m \in Ck0,\infty$is a bounded measurable function on$0,\infty$ such that

$$\left|\left(\frac{1}{y}\frac{d}{dy}\right)^{l} m(y)\right| \le y^{-s}(A\, y^{b-1})^{2l},\ 0 \le l \le k, \quad (4.2)$$

where $A \ge 1$ and $m(x) = 0$, $0 < x < \delta$, for certain $\delta > 0$. Then the Hankel multiplier M_m defines a bounded operator from $\mathcal{H}^{\infty}_{1,\alpha,\beta}$ into $L^1_{\alpha,\beta}$.

Proof: To see that M_m defines a bounded operator from $\mathcal{H}^{\infty}_{1,\alpha,\beta}$ into $L^1_{\alpha,\beta}$ it is sufficient to prove that there exists $C > 0$ such that

$$\|M_m b\|_1 \le C \quad (4.3)$$

for every $(1,\infty)$ −atom.

Indeed, let $f \in L^2_{\alpha,\beta} \cap \mathcal{H}^{\infty}_{1,\alpha,\beta}$. Assume that

$$f = \sum_{j=1}^{\infty} \lambda_j \tau_{x_j} b_j$$

in S'_e, where $\lambda_j \in \mathbb{C}$, $x_j \in (0,\infty)$ *and* b_j is an $(1,\infty)$ −atom , for every $j \in \mathbb{N}$, and being

$$\sum_{j=0}^{\infty} |\lambda_j| < \infty .$$

Then

$$M_m f = h_{\alpha,\beta}\left(m\, h_{\alpha,\beta}(f)\right) = \sum_{j=0}^{\infty} \lambda_j M_m\left(\tau_{x_j} b_j\right)$$

is in S'_e . Moreover, the last series converges in $L^1_{\alpha,\beta}$. Indeed, since, M_m commutes with Hankel type translations, from (4.3), it deduces

$$\sum_{j=n}^{l} |\lambda_j| \left\| M_m\left(\tau_{x_j} b_j\right)\right\|_1 \le C \sum_{j=n}^{l} |\lambda_j|, \quad n, l \in \mathbb{N}, \quad n > l.$$

Thus, as $L^1_{\alpha,\beta}$- convergence implies S'_e − convergence, we have

$$h_{\alpha,\beta}\left(m\, h_{\alpha,\beta}(f)\right)(x) = \sum_{j=0}^{\infty} \lambda_j\, M_m\left(\tau_{x_j}(a_j)\right)(x), \qquad a.e.\, x \in (0,\infty)$$

and

$$\|M_m f\|_1 \le C \sum_{j=0}^{\infty} |\lambda_j|.$$

Hence we conclude that

$$\|M_m f\|_1 \le C\, \|f\|^{\infty}_{1,\alpha,\beta} .$$

As $L^2_{\alpha,\beta} \cap \mathcal{H}^{\infty}_{1,\alpha,\beta}$ is a dense subspace of $\mathcal{H}^{\infty}_{1,\alpha,\beta}$, M_m can be extended to $\mathcal{H}^{\infty}_{1,\alpha,\beta}$ as a bounded operator from $\mathcal{H}^{\infty}_{1,\alpha,\beta}$ into $L^1_{\alpha,\beta}$.

We now prove (4.3). Suppose that $m(x) = 0,\ x \in (0,1)$. Otherwise we can proceed in a similar way. Let b be a $(1,\infty)$ −atom and assume that $b(x) = 0, x \geq a, \|b\|_\infty \leq \gamma\big((0,a)\big)^{-1}$. Since $\|b\|_2 \leq \gamma\big((0,a)\big)^{-1/2}$ *and* M_m is bounded from $L^2_{\alpha,\beta}$ into itself,

Holder's inequality leads to

$$\int_0^{2a} |M_m b(x)|\, d\gamma(x) \leq C\left\{\int_0^{2a} |M_m b(x)|^2\, d\gamma(x)\right\}^{1/2} a^{3\alpha+\beta} \leq C. \quad (4.4)$$

We choose a function $\phi \in C^\infty(0,\infty)$ such that $\phi(x) = 0, x \notin (1/2, 2)$ and

$$\sum_{j=-\infty}^{\infty} \phi\left(x/2^j\right) = 1,\ x \in (0,\infty)(see\ [19]).$$

Since $m(x) = 0,\ x \in (0,1)$

we can write

$$m(x) = \sum_{j=0}^{\infty} m_j(x),\ x \in (0,\infty),$$

where $m_j(x) = m(x)\,\phi\left(x/2^j\right), x \in (0,\infty)$ *and* $j \in \mathbb{N}$.

To simplify in the sequel we write M_j instead of M_{m_j}, $j \in \mathbb{N}$. Let $\in \mathbb{N}$ .Since $m_j \in L^2_{\alpha,\beta}$, we have that ([3, Lemma 2.1])

$$M_j b = k_j \# b\ , \text{ where } k_j = h_{\alpha,\beta}(m_j).$$

It is not hard to see that

$$\left|M_j b(x)\right| \leq \int_0^a \left|(\tau_x k_j)(y)\right| |b(y)|\, d\gamma(y) \leq \|b\|_\infty \int_0^a \left|(\tau_x k_j)(y)\right|\, d\gamma(y)$$

$$= C\, a^{-2(3\alpha+\beta)} \int_0^a \left|(\tau_x k_j)(y)\right| d\gamma(y),\ x \in (0,\infty). (4.5)$$

On the other hand, since

$$\int_0^a b(x)\, d\gamma(x) = 0,$$

according to $[23, p. 256]$ it has

$$M_j b(x) = \int_0^a b(y)\big(R_1(y) k_j\big)(x)\, d\gamma(y),\quad x \in (0,\infty), (4.6)$$

where for a measurable function f on $(0,\infty)$,

$$(R_1(y)f)(x) = \int_0^y \theta(y,\sigma)\tau_\sigma(\Delta_{\alpha,\beta}f)(x)\sigma^{4\alpha}\,d\sigma$$

being

$$\theta\,(y,\sigma) = \begin{cases} y^{-2(\alpha-\beta)} - \sigma^{-2(\alpha-\beta)}, & 0<\sigma<y \\ 0 \quad , & otherwise\,. \end{cases}$$

For every $l, s \in \mathbb{N},\ o \le l \le k,$ by (4.2), Leibniz's rule leads to

$$\left|\left(\frac{1}{x}\frac{d}{dx}\right)^l \left[x^{2s}m_j(x)\right]\right| \le C\sum_{i=0}^{l} 2^{-2j(l-i)+2sj} \underset{2^{j-1}\le x\le 2^{j+1}}{Sup} \left|\left(\frac{1}{x}\frac{d}{dx}\right)^i m_j(x)\right|$$

$$\le C\,2^{j(2s'-s)}\left(\gamma\left(2^{j-1},2^{j+1}\right)\right)^{1/2}$$

$$\le C\,2^{j(2s'-s+3\alpha+\beta)}, \qquad l,s\in\mathbb{N}, \qquad 0\le l\le k.$$

By taking into account now that

$$\Delta^i_{\alpha,\beta} = \sum_{h=0}^{i} C_{h,i}x^{2h}\left(\left(\frac{1}{x}\right)\left(\frac{d}{dx}\right)\right)^{i+h}, \ where \ \ C_{h,i}$$

is a suitable positive constant for every $h = 0,\ldots\ldots, i$ and $i \in \mathbb{N}$a straight-forward manipulation allows us to conclude

$$\left\|\left(1+A^{-2}2^{2j(1-b)}x^2\right)^l \Delta^s_{\alpha,\beta}k_j\right\|_2$$

$$\le C\sum_{i=0}^{l}\sum_{h=0}^{i} A^{-2i}2^{2j(1-b)i}\left\|x^{2h}\left(\frac{1}{x}\frac{d}{dx}\right)^{i+h}\left[x^{2s'}m_j(x)\right]\right\|_2$$

$$\le C\sum_{i=0}^{l}\sum_{h=0}^{i} A^{-2i}2^{2j(1-b)i+2jh}\left\|\left(\frac{1}{x}\frac{d}{dx}\right)^{i+h}\left[x^{2s'}m_j(x)\right]\right\|_2$$

$$\le C\,A^{2l}2^{j(2s'-s+3\alpha+\beta)}2^{2jbl},\ l,s\in\mathbb{N},\ 0\le l\le k. \quad (4.7)$$

By involving Holder's and Minkowski's inequalities [29, p.16] and (4.7) it follows

$$\int_{2a}^{\infty}\int_0^a \left|\left(\tau_x k_j\right)(y)\right| d\gamma(y)\,d\gamma(x)$$

$$\le \left\{\int_{2a}^{\infty}\left(\int_0^a \left|\left(\tau_x k_j\right)(y)\right| d\gamma(y)\left(1+A^{-2}2^{2j(1-b)}x^2\right)^k\right)^2 d\gamma(x)\right\}^{1/2}$$

$$\times \left\{ \int_0^\infty \left(1 + A^{-2} 2^{2j(1-b)} x^2\right)^{-2k} d\gamma(x) \right\}^{1/2}$$

$$\leq C \left(A^{-1} 2^{j(1-b)}\right)^{-(3\alpha+\beta)} \left\{ \int_{2a}^{\infty} \left(\int_0^a \int_{|x-y|}^{x+y} D_{\alpha,\beta}(x,y,z) |k_j(z)| d\gamma(z) d\gamma(y) \right.\right.$$

$$\left.\left. \times |k_j(z)| d\gamma(z) d\gamma(y) \left(1 + A^{-2} 2^{2j(1-b)} x^2\right)^k \right)^2 d\gamma(x) \right\}^{1/2}$$

$$\leq C \left(A^{-1} 2^{j(1-b)}\right)^{-(3\alpha+\beta)} \left\{ \int_{2a}^{a} \left(\int_0^a \int_{|x-y|}^{x+y} \left(1 + A^{-2} 2^{2j(1-b)} z^2\right)^k |k_j(z)| \right.\right.$$

$$\left.\left. \times D_{\alpha,\beta}(x,y,z) d\gamma(z) d\gamma(y) \right)^2 d\gamma(x) \right\}^{1/2}$$

$$= \left(A^{-1} 2^{j(1-b)}\right)^{-(3\alpha+\beta)} \left\{ \int_{2a}^{\infty} \left(\int_0^a \left[\tau_x \left(1 + A^{-2} 2^{2j(1-b)} z^2\right)^k |k_j(z)| \right] d\gamma(y) \right)^2 d\gamma(x) \right\}^{\frac{1}{2}}$$

$$\leq C \left(A^{-1} 2^{j(1-b)}\right)^{-(3\alpha+\beta)} \int_0^a \left\| \tau_y \left[\left(1 + A^{-2} 2^{2j(1-b)} z^2\right)^k |k_j| \right] \right\|_2 d\gamma(y)$$

$$\leq C \left(A^{-1} 2^{j(1-b)}\right)^{-(3\alpha+\beta)} a^{2(3\alpha+\beta)} \left\| \left(1 + A^{-2} 2^{2j(1-b)} z^2\right)^k k_j \right\|_2$$

$$\leq C \left(A^{-1} 2^{j(1-b)}\right)^{-(3\alpha+\beta)} A^{2k} 2^{j(3\alpha+\beta-s)} 2^{2jbk} a^{2(3\alpha+\beta)}$$

$$\leq C\, A^{2k+3\alpha+\beta} 2^{j(b(2k+3\alpha+\beta)-s)} a^{2(3\alpha+\beta)}. \quad (4.8)$$

By proceeding in a way similar to above (see [23, p.256]),

$$\int_{2a}^{\infty} \int_0^a \left| \left(R_1(y) k_j\right)(x) \right| d\gamma(y) d\gamma(x)$$

$$\leq C \left(A^{-1} 2^{j(1-b)}\right)^{-(3\alpha+\beta)} \int_0^a \int_0^y \left\| \tau_\sigma \left[\left(1 + A^{-2} 2^{2j(1-b)} z^2\right)^k |\Delta_{\alpha,\beta} k_j| \right] \right\|_2$$

$$\times\, \theta(y,\sigma)\, d\gamma(\sigma) d\gamma(y)$$

$$\leq C\left(A^{-1}2^{j(1-b)}\right)^{-(3\alpha+\beta)} A^{2k} 2^{j(2-s+3\alpha+\beta)} 2^{2jbk} \int_0^a \int_0^y \theta(y,\sigma)\, d\gamma(\sigma) d\gamma(y).$$

Since

$$\int_0^y \theta\,(y,\sigma) d\gamma(\sigma) \leq C\, y^2 \ , \ y \in (0,\infty)\,,$$

we conclude that

$$\int_{2a}^{\infty} \int_0^a \left|\left(R_1(y)k_j\right)(x)\right| d\gamma\,(y)\; d\gamma(x)$$

$$\leq C\, A^{2k+3\alpha+\beta} 2^{j(b(2k+3\alpha+\beta)-s+2)} a^{2(5\alpha+3\beta)}. \quad (4.9)$$

By combining (4.5), (4.6), (4.8) and (4.9), it obtains

$$\int_{2a}^{\infty} \left|M_j b(x)\right| d\gamma(x) \leq C\ A^{2k+3\alpha+\beta} 2^{j(a(2k+3\alpha+\beta)-s)}\,,$$

and

$$\int_{2a}^{\infty} \left|M_j b(x)\right|\, d\gamma\,(x) \leq C\, a^2 A^{2k+3\alpha+\beta} 2^{j(b(2k+3\alpha+\beta)-s+2)}.$$

Now, we choose $j_0 \in \mathbb{N}$ such that $2^{j_0}a \leq 1 < 2^{j_0+1}a$, provided that $a \leq 1$, and we take $j_0 = -1$, when $a > 1$. Since

$$\sum_{j=0}^{n} M_j b$$

converges to $M_m b$ as $n \to \infty$, in $L^2_{\alpha,\beta}$, we can write

$$\int_{2a}^{\infty} |M_m b(x)| d\gamma(x) \leq \sum_{j=0}^{\infty} \int_{2a}^{\infty} \left|M_j b(x)\right| d\gamma(x)$$

$$\leq C\left(\sum_{j=0}^{j_0} a^2 A^{2k+3\alpha+\beta}\, 2^{j(b(2k+3\alpha+\beta)-s+2)} + \sum_{j=j_0+1}^{\infty} A^{2k+3\alpha+\beta} 2^{j(b(2k+3\alpha+\beta)-s)}\right)$$

$$\leq C\left(a^2 A^{2k+3\alpha+\beta} 2^{j_0(b(2k+3\alpha+\beta)-s+2)} + A^{2k+3\alpha+\beta} 2^{j_0(b(2k+3\alpha+\beta)-s)}\right)$$

because $0 < s - b(2k+3\alpha+\beta) < 2$. Then, since $s > b\,(2k+3\alpha+\beta)$, it obtains

$$\int_{2a}^{\infty} |M_m b(x)|\, d\gamma(x) \leq C\ A^{2k+3\alpha+\beta}. \quad (4.10)$$

By combining (4.4) and (4.10), we obtain (4.3). Thus the proof is completed.

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