ON SOME PROPERTIES OF FUZZY SOFT SETS AND INTUITIONISTIC FUZZY SOFT SETS

Tanushree Mitra Basu*

Nirmal Kumar Mahapatra**

Shyamal Kumar Mondal*

Abstract:

The purpose of this paper is to prove some properties (i.e. associative, distributive) of Fuzzy Soft Sets and Intuitionistic Fuzzy Soft Sets with respect to extended union, extended intersection and restricted union, restricted intersection over the common universe. These properties are verified by appropriate examples.

Keywords: Fuzzy Soft Set(FSS), Intuitionistic Fuzzy Soft Set(IFSS), Associative Property, Distributive Property, extended intersection, restricted union, restricted intersection.

* Department of Applied Mathematics with Oceanology and Computer Programming, Vidyasagar University, Midnapore -721 102,W.B., India.

** Department of Mathematics, Panskumar Banamali College, Panskura RS-721152, WB, India.

1 Introduction:

Most of the recent mathematical methods meant for formal modelling,reasoning and computing are crisp, accurate and deterministic in nature. But in ground reality, crisp data is not always the part and parcel of the problems encountered in different fields like economics, engineering, social science, medical science, environment etc. As a consequence various theories viz. theory of probability, theory of fuzzy sets [2,7], theory of intuitionistic fuzzy sets [11,12], theory of vague sets [14], theory of interval mathematics [12,13], theory of rough sets [6] have been evolved in process. But difficulties present in all these theories have been shown by Molodtsov in [5]. The cause of these problems is possibly related to the inadequacy of the parametrization tool of the theories. As a result Molodtsov started the concept of soft theory [3,5,10,18] as a new mathematical tool for solving the uncertainties which is free from the above difficulties. Presence of vagueness demanded Fuzzy Soft Set [1,3,15] to come into picture. But satisfactory evaluation of membership values is not always possible because of the insufficiency in the available information (besides the presence of vagueness) situation. Evaluation of non-membership values is also not always possible for the same reason and as a result there exist an indeterministic part upon which hesitation survives. Certainly fuzzy soft set theory is not suitable to solve such problems. In those situations Intuitionistic Fuzzy Soft Set theory [16,17] may be more applicable. In our previous paper[19] we have given some new operations in FSS and IFSS Theory and proved that certain De Morgan's laws hold in FSS theory, as well as IFSS theory with respect to these new operations.In the present paper we establish some properties (associative, distributive) of FSS and IFSS with respect to these new operations.

2 Preliminaries:

We recall the definitions of Soft Set, Fuzzy Soft Set and Intuitionistic Fuzzy Soft Set with examples. Then we call back to mind, the extended, restricted union and the extended, restricted intersection property of Soft sets (or, FSS, or, IFSS) over the common universe.

2.1 Soft Sets: [18].

Let U be an initial universe set and E be a set of parameters. Let P(U) denotes the power set of U. Let $A \subset E$. A pair (F,A) is called a soft set over U, where F is a mapping given by, F : $A \to P(U)$

In other words, a soft set over U is a parameterized family of subsets of the universe U.

2.11 Example:

Let U be the set of four dresses, given by,

U = $\{d_1, d_2, d_3, d_4\}$

Let E be the set of parameters, given by,

E = $\{$ *costly, cheap, comfortable, beautiful, gorgeous* $\}$

Let $A \subset E$, given by,

A = $\{$ *costly, cheap, comfortable, beautiful* $\} = \{e_1, e_2, e_3, e_4\}$ where

e_1 stands for the parameter costly,

e_2 stands for the parameter cheap,

e_3 stands for the parameter comfortable,

e_4 stands for the parameter beautiful.

Now suppose that,F is a mapping, defined as dresses(.) and given by,

F(e_1) $= \{d_2, d_4\}$,

F(e_2) $= \{d_1, d_3\}$,

F(e_3) $= \{d_2, d_3\}$,

F(e_4) $= \{d_4\}$.

Then the Soft Set

$(F, A) = \{$ *costly dresses* $= \{d_2, d_4\}$,

cheap dresses $= \{d_1, d_3\}$,

comfortable dresses $= \{d_2, d_3\}$,

beautiful dresses $= \{d_4\}\}$

2.2 Fuzzy Soft Sets: [1].

Let U be an initial universe set and E be a set of parameters(which are fuzzy words or sentences involving fuzzy words). Let P(U) denotes the set of all fuzzy sets of U. Let $A \subset E$. A pair $(\tilde{F}, A)$ is called a fuzzy soft set (FSS) over U, where $\tilde{F}$ is a mapping given by, $\tilde{F} : A \to P(U)$.

2.21. Example:

Let U be the set of four cities, given by, U = $\{C_1, C_2, C_3, C_4\}$. Let E be the set of parameters (each parameter is a fuzzy word), given by,

E = $\{$ *highly, immensely, moderately, average, less* $\}$

Let $A \subset E$, given by,

A = $\{$ *highly, immensely, moderately, less* $\} = \{e_1, e_2, e_3, e_4\}$ where

e_1 stands for the parameter highly,

e_2 stands for the parameter immensely,

e_3 stands for the parameter moderately,

e_4 stands for the parameter less.

Now suppose that,

$\tilde{F}(e_1) = \{C_1/.2, C_2/.9, C_3/.4, C_4/.6\}$,

$\tilde{F}(e_2) = \{C_2/1, C_3/.3, C_4/.4\}$,

$\tilde{F}(e_3) = \{C_1/.3, C_2/.4, C_3/.8\}$,

$\tilde{F}(e_4) = \{C_1/.9, C_2/.1, C_3/.5, C_4/.3\}$

Then the Fuzzy Soft Set

$$(\tilde{F}, A) = \{highly\ polluted\ city = \{C_1/.2, C_2/.9, C_3/.4, C_4/.6\},$$
$$immensely\ polluted\ city = \{C_2/1, C_3/.3, C_4/.4\},$$
$$moderately\ polluted\ city = \{C_1/.3, C_2/.4, C_3/.8\},$$
$$less\ polluted\ city = \{C_1/.9, C_2/.1, C_3/.5, C_4/.3\}\}$$

2.3 Intuitionistic Fuzzy Soft Sets: [17].

Let U be an initial universe set and E be a set of parameters. Let P(U) denotes the set of all intuitionistic fuzzy sets of U. Let $A \subset E$. A pair $(\hat{F}, A)$ is called an intuitionistic fuzzy soft set(IFSS) over U, where $\hat{F}$ is a mapping given by, $\hat{F} : A \to P(U)$.

2.31. Example:

Let U be the set of four cities, given by, U = $\{C_1, C_2, C_3, C_4\}$. Let E be the set of parameters (each parameter is a intuitionistic fuzzy word), given by,

E = $\{ highly, immensely, moderately, average, less \}$.

Let $A \subset E$, given by,

A = $\{ highly, immensely, moderately, less \}$

= $\{e_1, e_2, e_3, e_4\}$

where e_1 stands for the parameter highly,

e_2 stands for the parameter immensely,

e_3 stands for the parameter moderately,

e_4 stands for the parameter less.

Now suppose that,

$\hat{F}(e_1) = \{C_1/(.2,.7), C_2/(.8,.1), C_3/(.4,.2), C_4/(.6,.3)\}$,

$\hat{F}(e_2) = \{C_2/(.9,.1), C_3/(.3,.6), C_4/(.4,.6)\}$,

$\hat{F}(e_3) = \{C_1/(.3,.5), C_2/(.4,.6), C_3/(.8,.1)\}$,

$\hat{F}(e_4) = \{C_1/(.9,.1), C_2/(.1,.8), C_3/(.5,.4), C_4/(.3,.5)\}$

Then the Intuitionistic Fuzzy Soft Set

$$(\hat{F}, A) = \{highly\ polluted\ city = \{C_1/(.2,.7), C_2/(.8,.1), C_3/(.4,.2), C_4/(.6,.3)\},$$
$$immensely\ polluted\ city = \{C_2/(.9,.1), C_3/(.3,.6), C_4/(.4,.6)\},$$
$$moderately\ polluted\ city = \{C_1/(.3,.5), C_2/(.4,.6), C_3/(.8,.1)\},$$

$$less\ polluted\ city = \{C_1/(.9,.1), C_2/(.1,.8), C_3/(.5,.4), C_4/(.3,.5)\}\}$$

Now we recall some properties of FSS and IFSS.

2.4 Extended Union of two FSS (or, IFSS): [19]

The extended union of two fuzzy soft sets $(\tilde{F}, A)$ and $(\tilde{G}, B)$ over a common universe U is the FSS $(\tilde{H}, C)$, where C = $A \cup B$ and $\forall e \varepsilon C$,

$$\tilde{H}(e) = \begin{cases} \tilde{F}(e), & if\ e\varepsilon(A-B) \\ \tilde{G}(e), & if\ e\varepsilon(B-A) \\ \tilde{F}(e)\tilde{\cup}\tilde{G}(e), & if\ e\varepsilon A\cap B \end{cases}$$

We write,

$((\tilde{F}, A)\tilde{\cup}_E (\tilde{G}, B)) = (\tilde{H}, C)$ [for FSS] and

$((\hat{F}, A)\hat{\cup}_E (\hat{G}, B)) = (\hat{H}, C)$ [for IFSS]

2.5 Restricted Union of two FSS (or, IFSS):[19]

Let $(\tilde{F}, A)$ and $(\tilde{G}, B)$ be two fuzzy soft sets over a common universe U such that $A \cap B \neq \phi$ (null set). The restricted union of $(\tilde{F}, A)$ and $(\tilde{G}, B)$ is denoted by $((\tilde{F}, A)\tilde{\cup}_R (\tilde{G}, B))$ [or, $((\hat{F}, A)\hat{\cup}_R (\hat{G}, B))$ for IFSS] and is defined as,

$((\tilde{F}, A)\tilde{\cup}_R (\tilde{G}, B)) = (\tilde{H}, C)$, where C = $A \cap B$ and

$\forall c\varepsilon C, \tilde{H}(c) = \tilde{F}(c)\tilde{\cup}\tilde{G}(c)$ [where $\tilde{\cup}$ is the operation fuzzy union of two fuzzy sets.]

2.6 Extended Intersection of two FSS (or, IFSS): [19]

The extended intersection of two fuzzy soft sets $(\tilde{F}, A)$ and $(\tilde{G}, B)$ over a common universe U is the FSS $(\tilde{H}, C)$, where C = $A \cup B$ and $\forall e \varepsilon C$,

$$\tilde{H}(e) = \begin{cases} \tilde{F}(e), & if\ e\varepsilon(A-B) \\ \tilde{G}(e), & if\ e\varepsilon(B-A) \\ \tilde{F}(e)\tilde{\cap}\tilde{G}(e), & if\ e\varepsilon A\cap B \end{cases}$$

We write,

$((\tilde{F}, A)\tilde{\cap}_E (\tilde{G}, B)) = (\tilde{H}, C)$ [for FSS] and

$((\hat{F}, A)\hat{\cap}_E (\hat{G}, B)) = (\hat{H}, C)$ [for IFSS]

2.7 Restricted Intersection of two FSS (or, IFSS):[19]

Let $(\tilde{F},A)$ and $(\tilde{G},B)$ be two fuzzy soft sets over a common universe U such that $A\cap B\neq\phi$ (null set). The restricted intersection of $(\tilde{F},A)$ and $(\tilde{G},B)$ is denoted by $((\tilde{F},A)\tilde{\cap}_R(\tilde{G},B))$ [or, $((\hat{F},A)\hat{\cap}_R(\hat{G},B))$ for IFSS] and is defined as,

$((\tilde{F},A)\tilde{\cap}_R(\tilde{G},B))=(\tilde{H},C)$, where C = $A\cap B$ and

$\forall c\varepsilon C, \tilde{H}(c)=\tilde{F}(c)\tilde{\cap}\tilde{G}(c)$ [where $\tilde{\cap}$ is the operation fuzzy intersection of two fuzzy sets.]

Properties:

Throughout this paper U refers to an initial universe, P(U) is the power set of U. A, B, C are sets of parameters. $\tilde{\cap}[or,\hat{\cap}]$ denotes the operation fuzzy intersection[or, intuitionistic fuzzy intersection] and $\tilde{\cup}[or,\hat{\cap}]$ denotes the operation fuzzy union[or, intuitionistic fuzzy union].

3 Associative Property of Fuzzy Soft Sets and Intuitionistic Fuzzy Soft Sets w.r.t. extended union over the same universe:

Let $(\tilde{F},A)$, $(\tilde{G},B)$, $(\tilde{H},C)$ are three FSS (or, IFSS). Then we have to prove,

$(\tilde{F},A)\tilde{\cup}_E((\tilde{G},B)\tilde{\cup}_E(\tilde{H},C))=((\tilde{F},A)\tilde{\cup}_E(\tilde{G},B))\tilde{\cup}_E(\tilde{H},C)$

Proof:

$$L.H.S=(\tilde{F},A)\tilde{\cup}_E(\tilde{O},D) \text{ where } D=B\cup C \text{ and } \tilde{O}(e)=\begin{cases}\tilde{G}(e), & if\ e\varepsilon(B-C)\\ \tilde{H}(e), & if\ e\varepsilon(C-B)\\ \tilde{G}(e)\tilde{\cup}\tilde{H}(e), & if\ e\varepsilon B\cap C\end{cases}$$

$$=(\tilde{L},E)$$

where E = $A\cup D=A\cup(B\cup C)$ and

$$\tilde{L}(e)=\begin{cases}\tilde{F}(e), & if\ e\varepsilon A-(B\cup C)\ ie., if\ e\varepsilon(A-B)\cap(A-C)\\ \tilde{O}(e), & if\ e\varepsilon(D-A)\ ie., if\ e\varepsilon(B-A)\cup(C-A)\\ \tilde{F}(e)\tilde{\cup}O(e), & if\ e\varepsilon A\cap D\ ie., if\ e\varepsilon A\cap(B\cup C)\end{cases}$$

$$\text{ie., } \tilde{L}(e)=\begin{cases}\tilde{F}(e), & if\ e\varepsilon(A-B)\cap(A-C)\\ \tilde{G}(e), & if\ e\varepsilon(B-C)\cap(B-A)\\ \tilde{H}(e), & if\ e\varepsilon(C-B)\cap(C-A)\\ \tilde{G}(e)\tilde{\cup}\tilde{H}(e), & if\ e\varepsilon(B-A)\cap(C-A)\\ \tilde{F}(e)\tilde{\cup}\tilde{G}(e), & if\ e\varepsilon(B-C)\cap(A-C)\\ \tilde{F}(e)\tilde{\cup}\tilde{H}(e), & if\ e\varepsilon(A-B)\cap(C-B)\\ \tilde{F}(e)\tilde{\cup}(\tilde{G}(e)\tilde{\cup}\tilde{H}(e)), & if\ e\varepsilon A\cap B\cap C\end{cases}$$

$$R.H.S = ((\tilde{F},A)\tilde{\cup}_E(\tilde{G},B))\tilde{\cup}_E(\tilde{H},C)$$

$$= (\tilde{I},K)\tilde{\cup}_E(\tilde{H},C)[\ where\ K = A\cup B\ and\ \tilde{I}(e) = \begin{cases}\tilde{F}(e), & if\ e\varepsilon(A-B)\\ \tilde{G}(e), & if\ e\varepsilon(B-A)]\\ \tilde{F}(e)\tilde{\cup}\tilde{G}(e), & if\ e\varepsilon A\cap B\end{cases}$$

$$= (\tilde{J},M)\ where M = K\cup C = (A\cup B)\cup C = A\cup(B\cup C)$$

$[\ by\ the\ associative\ property of\ crisp\ sets\ ]\ and$

$$\tilde{J}(e) = \begin{cases}\tilde{I}(e), & if\ e\varepsilon(K-C),\ ie.,\ if\ e\varepsilon(A-C)\cup(B-C)\\ \tilde{H}(e), & if\ e\varepsilon(C-K),\ ie.,\ if\ e\varepsilon(C-A)\cup(C-B)\\ \tilde{I}(e)\tilde{\cup}\tilde{H}(e), & if\ e\varepsilon K\cap C,\ ie.,\ if\ e\varepsilon(A\cup B)\cap C\end{cases}$$

ie.,

$$\tilde{J}(e) = \begin{cases}\tilde{F}(e), & if\ e\varepsilon(A-B)\cap(A-C)\\ \tilde{G}(e), & if\ e\varepsilon(B-A)\cap(B-C)\\ \tilde{F}(e)\tilde{\cup}\tilde{G}(e), & if\ e\varepsilon(A-C)\cap(B-C)\\ \tilde{H}(e), & if\ e\varepsilon(C-A)\cap(C-B)\\ \tilde{F}(e)\tilde{\cup}\tilde{H}(e), & if\ e\varepsilon(A-B)\cap(C-B)\\ \tilde{G}(e)\tilde{\cup}\tilde{H}(e), & if\ e\varepsilon(B-A)\cap(C-A)\\ (\tilde{F}(e)\tilde{\cup}\tilde{G}(e))\tilde{\cup}\tilde{H}(e)\ ie\ \tilde{F}(e)\tilde{\cup}(\tilde{G}(e)\tilde{\cup}\tilde{H}(e)), & if\ e\varepsilon A\cap B\cap C\\ & (by\ associative\ property of\ fuzzy\ sets)\end{cases}$$

Now $(\tilde{L},E)$ and $(\tilde{J},M)$ are two fuzzy soft sets (or, intuitionistic fuzzy soft set) over the common universe, where

i) $E\subset M$ and

ii) $\forall e\varepsilon E, \tilde{L}(e)$ is a fuzzy subset (or, IF subset) of $\tilde{J}(e)$.

$\Rightarrow(\tilde{L},E)$ is a fuzzy soft subset (or, intuitionistic fuzzy soft subset) of $(\tilde{J},M)$. (1)

Again, i) $M\subset E$ and

ii) $\forall e\varepsilon M, \tilde{J}(e)$ is a fuzzy subset (or, IF subset) of $\tilde{L}(e)$

$\Rightarrow(\tilde{J},M)$ is a fuzzy soft subset (or IF soft subset) of $(\tilde{L},E)$. (2)

Combining (1) and (2),we can conclude that, $(\tilde{L},E)$ and $(\tilde{J},M)$ are equal FSS (or, IFSS) over the common universe. Hence proved.

Verification of this property by examples:

3.1 Example for FSS

Let U be the set of four factories, say, U = $\{F_1, F_2, F_3, F_4\}$. Let A,B,C be three sets of parameters, given by,

A = { *good*, *poor* }

B = { *assured, good* }

C = { *assured, good, poor* }

Now let $(\tilde{F}, A)$ denotes work culture of the factory,

$(\tilde{G}, B)$ denotes production of the factory,

$(\tilde{H}, C)$ denotes quality of the products of the factory

given by,

$\tilde{F}$ (good) = $\{F_1/.8, F_2/.2, F_3/.5, F_4/.4\}$

$\tilde{F}$ (poor) = $\{F_1/.2, F_2/.7, F_3/.4, F_4/.6\}$

$\tilde{G}$ (assured) = $\{F_1/.9, F_2/.3, F_3/.6, F_4/.5\}$

$\tilde{G}$ (good) = $\{F_1/.7, F_2/.2, F_3/.4, F_4/.5\}$

$\tilde{H}$ (assured) = $\{F_1/.8, F_2/.3, F_3/.4, F_4/.4\}$

$\tilde{H}$ (good) = $\{F_1/.8, F_2/.3, F_3/.4, F_4/.4\}$

$\tilde{H}$ (poor) = $\{F_1/.1, F_2/.6, F_3/.3, F_4/.6\}$

Now consider,

$$(\tilde{F}, A)\,\tilde{\cup}_E\,((\tilde{G}, B)\,\tilde{\cup}_E\,(\tilde{H}, C))$$

$$= (\tilde{F}, A)\,\tilde{\cup}_E\,(\tilde{O}, D)[\;where\; D = B \cup C = \{\,assured, good, poor\,\}$$

$$and\;\tilde{O}(assured) = \{F_1/.9, F_2/.3, F_3/.6, F_4/.5\},$$

$$\tilde{O}(\;good\;) = \{F_1/.8, F_2/.3, F_3/.4, F_4/.5\}$$

$$\tilde{O}(\;poor) = \{F_1/.1, F_2/.6, F_3/.3, F_4/.6\}]$$

$$= (\tilde{L}, E)\;where\; E = A \cup D = \{\,assured, good, poor\,\}$$

$$and\;\tilde{L}(assured) = \{F_1/.9, F_2/.3, F_3/.6, F_4/.5\},$$

$$\tilde{L}(\;good) = \{F_1/.8, F_2/.3, F_3/.5, F_4/.5\},$$

$$\tilde{L}(\;poor) = \{F_1/.2, F_2/.7, F_3/.4, F_4/.6\}$$

Next consider,

$$((\tilde{F}, A)\,\tilde{\cup}_E\,(\tilde{G}, B))\,\tilde{\cup}_E\,(\tilde{H}, C)$$

$$= (\tilde{I}, K)\,\tilde{\cup}_E\,(\tilde{H}, C)[\;where\; K = A \cup B = \{\,assured, good, poor\,\}$$

$$and\;\tilde{I}(assured\,) = \{F_1/.9, F_2/.3, F_3/.6, F_4/.5\},$$

$$\tilde{I}(\;good) = \{F_1/.8, F_2/.2, F_3/.5, F_4/.5\},$$

$$\tilde{I}(poor\,) = \{F_1/.2, F_2/.7, F_3/.4, F_4/.6\}]$$

$$= (\tilde{J}, M)[\;where\; M = K \cup C = \{\,assured, good, poor\,\}$$

$$and\;\tilde{J}(\;assured) = \{F_1/.9, F_2/.3, F_3/.6, F_4/.5\}$$

$$\tilde{J}(good\,) = \{F_1/.8, F_2/.3, F_3/.5, F_4/.5\}$$

$$\tilde{J}(poor) = \{F_1/.2, F_2/.7, F_3/.4, F_4/.6\}]$$

Here $(\tilde{L},E)$ and $(\tilde{J},M)$ are two FSS which satisfy the following properties,

i) $E \subset M$ and

ii) $\forall e \varepsilon E, \tilde{L}(e)$ is a fuzzy subset of $\tilde{J}(e)$.

$\Rightarrow (\tilde{L},E) \tilde{\subseteq} (\tilde{J},M)$ (3)

Similarly, $(\tilde{J},M) \tilde{\subseteq} (\tilde{L},E)$ (4)

Combining (3) and (4) we have, $(\tilde{L},E)$ and $(\tilde{J},M)$ are fuzzy soft equal. Hence the associative property of FSS w.r.t extended union over the common universe is verified by this example.

3.2 Example for IFSS.

Let U be the set of four cities, say, U = $\{C_1, C_2, C_3, C_4\}$. Let A, B, C be three sets of parameters,given by,

A = $\{ huge, average, low \}$

B = $\{ large, average \}$

C = $\{ huge, average \}$

Now let $(\hat{F}, A)$ denotes population of the cities,

$(\hat{G}, B)$ denotes area of the cities,

$(\hat{H}, C)$ denotes pollution in the cities

given by,

$\hat{F}$ (huge) = $\{C_1/(.8,.1), C_2/(.3,.5), C_3/(1,0), C_4/(.6,.2)\}$

$\hat{F}$ (average) = $\{C_1/(.6,.3), C_2/(.8,.2), C_3/(.2,.4), C_4/(.4,.2)\}$

$\hat{F}$ (low) = $\{C_1/(.2,.6), C_2/(.7,.2), C_3/(0,1), C_4/(.4,.3)\}$

$\hat{G}$ (large) = $\{C_1/(.4,.3), C_2/(.8,.1), C_3/(.9,.1), C_4/(.5,.2)\}$

$\hat{G}$ (average) = $\{C_1/(.7,.2), C_2/(.5,.3), C_3/(.1,.5), C_4/(.6,.3)\}$

$\hat{H}$ (huge) = $\{C_1/(.7,.1), C_2/(.4,.3), C_3/(.8,.1), C_4/(.5,.3)\}$

$\hat{H}$ (average) = $\{C_1/(.6,.2), C_2/(.7,.2), C_3/(.3,.6), C_4/(.5,.4)\}$

Now consider,

$$(\hat{F}, A) \hat{\cup}_E ((\hat{G}, B) \hat{\cup}_E (\hat{H}, C))$$

$$= (\hat{F}, A) \hat{\cup}_E (\hat{O}, D)[\ where\ D = B \cup C = \{ large, average, huge \}$$

$$and\ \hat{O}(large) = \{C_1/(.4,.3), C_2/(.8,.1), C_3/(.9,.1), C_4/(.5,.2)\},$$

$$\hat{O}(average) = \{C_1/(.7,.2), C_2/(.7,.2), C_3/(.3,.5), C_4/(.6,.3)\},$$

$$\hat{O}(huge) = \{C_1/(.7,.1), C_2/(.4,.3), C_3/(.8,.1), C_4/(.5,.3)\}]$$

$$= (\hat{L}, E)[\ where\ E = A \cup D = A \cup (B \cup C) = \{ huge, average, low, large \}$$

$$and\ \hat{L}(\ huge) = \{C_1/(.8,.1), C_2/(.4,.3), C_3/(1,0), C_4/(.6,.2)\}$$

$$\hat{L}(\ average) = \{C_1/(.7,.2), C_2/(.8,.2), C_3/(.3,.4), C_4/(.6,.2)\}$$

$$\hat{L}(low) = \{C_1/(.2,.6), C_2/(.7,.2), C_3/(0,1), C_4/(.4,.3)\}]$$

$\hat{L}(large) = \{C_1/(.4,.3), C_2/(.8,.1), C_3/(.9,.1), C_4/(.5,.2)\}]$

Next consider,

$$((\hat{F},A)\hat{\cup}_E(\hat{G},B))\hat{\cup}_E(\hat{H},C)$$
$$=(\hat{I},K)\hat{\cup}_E(\hat{H},C)[\ where\ K = A\cup B = \{huge, average, low\}$$
$$and\ \hat{I}(huge) = \{C_1/(.8,.1), C_2/(.3,.5), C_3/(1,0), C_4/(.6,.2)\}$$
$$\hat{I}(average) = \{C_1/(.7,.2), C_2/(.8,.2), C_3/(.2,.4), C_4/(.6,.2)\}$$
$$\hat{I}(low) = \{C_1/(.2,.6), C_2/(.7,.2), C_3/(0,1), C_4/(.4,.3)\}$$
$$\hat{I}(large) = \{C_1/(.4,.3), C_2/(.8,.1), C_3/(.9,.1), C_4/(.5,.2)\}$$
$$=(\hat{J},M)[\ where\ M = K\cup C = (A\cup B)\cup C = \{huge, average, low\}$$
$$and\ \hat{J}(huge) = \{C_1/(.8,.1), C_2/(.4,.3), C_3/(1,0), C_4/(.6,.2)\}$$
$$\hat{J}(average) = \{C_1/(.7,.2), C_2/(.8,.2), C_3/(.3,.4), C_4/(.6,.2)\}$$
$$\hat{J}(low) = \{C_1/(.2,.6), C_2/(.7,.2), C_3/(0,1), C_4/(.4,.3)\}]$$
$$\hat{J}(large) = \{C_1/(.4,.3), C_2/(.8,.1), C_3/(.9,.1), C_4/(.5,.2)\}$$

Here $(\hat{L},E)$ and $(\hat{J},M)$ are two IFSS over the common universe where

i) $E \subset M$

ii) $\forall e\varepsilon E, \hat{L}(e)$ is a intuitionistic fuzzy subset of $\hat{J}(e)$. (5)

$[\tilde{\subseteq}\ denotes\ intuitionistic\ fuzzy\ soft\ subset]$

Similarly, $(\hat{J},M)\tilde{\subseteq}(\hat{L},E)$ (6)

Combining (5) and (6) we conclude that, $(\hat{L},E)$ and $(\hat{J},M)$ are IFS equal. Hence the associative property of IFSS w.r.t extended union over the common universe is verified by this example.

4 Associative Property of Fuzzy Soft Sets and Intuitionistic Fuzzy Soft Sets w.r.t. restricted union over the same universe:

Let $(\tilde{F},A)$, $(\tilde{G},B)$, $(\tilde{H},C)$ are three FSS (or, IFSS). Then we have to prove,

$(\tilde{F},A)\tilde{\cup}_R((\tilde{G},B)\tilde{\cup}_R(\tilde{H},C)) = ((\tilde{F},A)\tilde{\cup}_R(\tilde{G},B))\tilde{\cup}_R(\tilde{H},C)$

Proof:

$$L.H.S = (\tilde{F},A)\tilde{\cup}_R(\tilde{O},D)[\ where\ D = B\cap C\ and\ \forall d\varepsilon D,$$
$$\tilde{O}(d) = \tilde{G}(d)\tilde{\cup}\tilde{H}(d)(where\ \tilde{\cup}\ is\ the\ operation\ fuzzy\ union\ of\ two\ fuzzy\ sets.)]$$
$$=(\tilde{L},E)$$

where E = $A\cap D = A\cap(B\cap C)$ and $\forall e\varepsilon A\cap(B\cap C)$

$$\tilde{L}(e) = \tilde{F}(e)\tilde{\cup}O(e)$$
$$= \tilde{F}(e)\tilde{\cup}(\tilde{G}(e)\tilde{\cup}\tilde{H}(e))$$

$$R.H.S = ((\tilde{F},A)\tilde{\cup}_R(\tilde{G},B))\tilde{\cup}_R(\tilde{H},C)$$
$$= (\tilde{I},K)\tilde{\cup}_R(\tilde{H},C)[\ where\ K = A\cap B\ and\ I(e) = \tilde{F}(e)\tilde{\cup}\tilde{G}(e)\forall e\varepsilon A\cap B$$
$$= (\tilde{J},M) where M = K\cap C = (A\cap B)\cap C = A\cap(B\cap C)$$
$$[\ by\ the\ associative\ property of\ crisp\ sets\ ]$$

and $\forall e\varepsilon A\cap(B\cap C)$

$$\tilde{J}(e) = \tilde{I}(e)\tilde{\cup}H(e)$$
$$= (\tilde{F}(e)\tilde{\cup}\tilde{G}(e))\tilde{\cup}\tilde{H}(e)$$
$$= \tilde{F}(e)\tilde{\cup}(\tilde{G}(e)\tilde{\cup}\tilde{H}(e))$$
$$[\ by\ the\ associative\ property of\ fuzzy sets\ (\ or,\ intuitionistic\ fuzzy sets\ )\ ]$$

Now $(\tilde{L},E)$ and $(\tilde{J},M)$ are two fuzzy soft sets (or, intuitionistic fuzzy soft set) over the common universe,where

i) $E\subset M$ and

ii) $\forall e\varepsilon E, \tilde{L}(e)$ is a fuzzy subset (or, IF subset) of $\tilde{J}(e)$.

$\Rightarrow(\tilde{L},E)$ is a fuzzy soft subset (or, intuitionistic fuzzy soft subset) of $(\tilde{J},M)$. (7)

Again, (i) $M\subset E$ and

(ii) $\forall e\varepsilon M, \tilde{J}(e)$ is a fuzzy subset (or, IF subset) of $\tilde{L}(e)$.

$\Rightarrow(\tilde{J},M)$ is a fuzzy soft subset (or IF soft subset) of $(\tilde{L},E)$. (8)

Combining (7) and (8),we can conclude that, $(\tilde{L},E)$ and $(\tilde{J},M)$ are equal FSS (or, IFSS) over the common universe. Hence proved.

4.1 Example for FSS

Consider example 3.1. Now

$$(\tilde{F},A)\tilde{\cup}_R((\tilde{G},B)\tilde{\cup}_R(\tilde{H},C))$$
$$= (\tilde{F},A)\tilde{\cup}_R(\tilde{O},D)[\ where\ D = B\cap C = \{\ assured,\ good\ \}$$
$$and\ \tilde{O}(assured) = \{F_1/.9, F_2/.3, F_3/.6, F_4/.5\},$$
$$\tilde{O}(\ good\ ) = \{F_1/.8, F_2/.3, F_3/.4, F_4/.5\}$$
$$= (\tilde{L},E)\ where\ E = A\cap D = \{\ good\ \}$$
$$and\ \tilde{L}(\ good) = \{F_1/.8, F_2/.3, F_3/.5, F_4/.5\}$$

Next

$$((\tilde{F},A)\tilde{\cup}_R(\tilde{G},B))\tilde{\cup}_R(\tilde{H},C)$$
$$=(\tilde{I},K)\tilde{\cup}_R(\tilde{H},C)[\ where\ K=A\cap B=\{\ good\ \}$$
$$and\ \tilde{I}(\ good)=\{F_1/.8,F_2/.2,F_3/.5,F_4/.5\},$$
$$=(\tilde{J},M)[\ where\ M=K\cap C=\{\ good\ \}$$
$$and\ \tilde{J}(good\ )=\{F_1/.8,F_2/.3,F_3/.5,F_4/.5\}]$$

Here $(\tilde{L},E)$ and $(\tilde{J},M)$ are two FSS which satisfy the following properties,

i) $E\subset M$ and

ii) $\forall e\varepsilon E,\tilde{L}(e)$ is a fuzzy subset of $\tilde{J}(e)$.

$\Rightarrow(\tilde{L},E)\tilde{\subseteq}(\tilde{J},M)$ (9)

Similarly, $(\tilde{J},M)\tilde{\subseteq}(\tilde{L},E)$ (10)

Combining (9) and (10) we have, $(\tilde{L},E)$ and $(\tilde{J},M)$ are fuzzy soft equal. Hence the associative property of FSS w.r.t restricted union over the common universe is verified by this example.

4.2 Example for IFSS

Consider Example 3.2. Now

$$(\hat{F},A)\hat{\cup}_R((\hat{G},B)\hat{\cup}_R(\hat{H},C))$$
$$=(\hat{F},A)\hat{\cup}_R(\hat{O},D)[\ where\ D=B\cap C=\{\ huge\}$$
$$and\ \hat{O}(\ huge\ )=\{C_1/(.7,.1),C_2/(.4,.3),C_3/(.8,.1),C_4/(.5,.3)\}]$$
$$=(\hat{L},E)[\ where\ E=A\cap D=A\cap(B\cap C)=\{\ huge\}$$
$$and\ \hat{L}(\ huge)=\{C_1/(.8,.1),C_2/(.4,.3),C_3/(1,0),C_4/(.6,.2)\}]$$

Next

$$((\hat{F},A)\hat{\cup}_R(\hat{G},B))\hat{\cup}_R(\hat{H},C)$$
$$=(\hat{I},K)\hat{\cup}_R(\hat{H},C)[\ where\ K=A\cap B=\{\ huge,average\}$$
$$and\ \hat{I}(\ huge\ )=\{C_1/(.8,.1),C_2/(.3,.5),C_3/(1,0),C_4/(.6,.2)\}$$
$$\hat{I}(\ average)=\{C_1/(.7,.2),C_2/(.8,.2),C_3/(.2,.4),C_4/(.6,.2)\}]$$
$$=(\hat{J},M)[\ where\ M=K\cap C=(A\cap B)\cap C=\{\ huge\}$$
$$and\ \hat{J}(\ huge)=\{C_1/(.8,.1),C_2/(.4,.3),C_3/(1,0),C_4/(.6,.2)\}]$$

Here $(\hat{L},E)$ and $(\hat{J},M)$ are two IFSS over the common universe where

i) $E\subset M$

ii) $\forall e\varepsilon E,\hat{L}(e)$ is a intuitionistic fuzzy subset of $\hat{J}(e)$.

$\Rightarrow(\hat{L},E)\hat{\subseteq}(\hat{J},M)$ (11)

[$\hat{\subseteq}$ denotes intuitionistic fuzzy soft subset]

Similarly, $(\hat{J},M)\hat{\subseteq}(\hat{L},E)$ (12)

Combining (11) and (12) we conclude that, $(\hat{L}, E)$ and $(\hat{J}, M)$ are IFS equal. Hence the associative property of IFSS w.r.t restricted union over the common universe is verified by this example.

5 Associative property of FSS (or, IFSS) w.r.t extended intersection over the common universe:

Let (F, A), (G, B), (H, C) are three FSS (or, IFSS). Then we have to prove, $(F,A)\tilde{\cap}_E ((G,B)\tilde{\cap}_E (H,C)) = ((F,A)\tilde{\cap}_E (G,B))\tilde{\cap}_E (H,C)$

Proof:

$$L.H.S = (F,A)\tilde{\cap}_E ((G,B)\tilde{\cap}_E (H,C))$$

$$= (F,A)\tilde{\cap}_E (O,D)[\ where\ D = B\cup C\ and\ \tilde{O}(e) = \begin{cases} \tilde{G}(e), & if\ e\varepsilon(B-C) \\ \tilde{H}(e), & if\ e\varepsilon(C-B) \\ \tilde{G}(e)\tilde{\cap}\tilde{H}(e), & if\ e\varepsilon B\cap C \end{cases}$$

$(where\ \tilde{\cap}\ is\ the\ operation\ fuzzy\ intersection\ of\ two\ fuzzy\ sets.)]$

$$= (\tilde{L}, E)$$

where $E = A\cup D = A\cup(B\cup C) = (A\cup B)\cup C$ [by associative property of crisp set] and

$$\tilde{L}(e) = \begin{cases} \tilde{F}(e), & if\ e\varepsilon A-(B\cup C)\ ie.,\ if\ e\varepsilon(A-B)\cap(A-C) \\ \tilde{O}(e), & if\ e\varepsilon(D-A)\ ie.,\ if\ e\varepsilon(B-A)\cup(C-A) \\ \tilde{F}(e)\tilde{\cap}O(e), & if\ e\varepsilon A\cap D\ ie.,\ if\ e\varepsilon A\cap(B\cup C) \end{cases}$$

ie., $$\tilde{L}(e) = \begin{cases} \tilde{F}(e), & if\ e\varepsilon(A-B)\cap(A-C) \\ \tilde{G}(e), & if\ e\varepsilon(B-C)\cap(B-A) \\ \tilde{H}(e), & if\ e\varepsilon(C-B)\cap(C-A) \\ \tilde{G}(e)\tilde{\cap}\tilde{H}(e), & if\ e\varepsilon(B-A)\cap(C-A) \\ \tilde{F}(e)\tilde{\cap}\tilde{G}(e), & if\ e\varepsilon(B-C)\cap(A-C) \\ \tilde{F}(e)\tilde{\cap}\tilde{H}(e), & if\ e\varepsilon(A-B)\cap(C-B) \\ \tilde{F}(e)\tilde{\cap}(\tilde{G}(e)\tilde{\cap}\tilde{H}(e)), & if\ e\varepsilon A\cap B\cap C \end{cases}$$

$$R.H.S. = ((\tilde{F},A)\tilde{\cap}_E (\tilde{G},B))\tilde{\cap}_E (\tilde{H},C)$$

$$= (\tilde{I},K)\tilde{\cap}_E (\tilde{H},C)[\ where\ K = A\cup B\ and\ \tilde{I}(e) = \begin{cases} \tilde{F}(e), & if\ e\varepsilon(A-B) \\ \tilde{G}(e), & if\ e\varepsilon(B-A)] \\ \tilde{F}(e)\tilde{\cap}\tilde{G}(e), & if\ e\varepsilon A\cap B \end{cases}$$

$$= (\tilde{J}, M) where M = K \cup C = (A \cup B) \cup C \ and$$

$$\tilde{J}(e) = \begin{cases} \tilde{I}(e), & if \ e\varepsilon(K-C), ie., if \ e\varepsilon(A-C) \cup (B-C) \\ \tilde{H}(e), & if \ e\varepsilon(C-K), ie., if \ e\varepsilon(C-A) \cup (C-B) \\ \tilde{I}(e) \tilde{\cap} \tilde{H}(e), & if \ e\varepsilon K \cap C, ie., if \ e\varepsilon(A \cup B) \cap C \end{cases}$$

$$\text{ie., } \tilde{J}(e) = \begin{cases} \tilde{F}(e), & if \ e\varepsilon(A-B) \cap (A-C) \\ \tilde{G}(e), & if \ e\varepsilon(B-A) \cap (B-C) \\ \tilde{F}(e) \tilde{\cap} \tilde{G}(e), & if \ e\varepsilon(A-C) \cap (B-C) \\ \tilde{H}(e), & if \ e\varepsilon(C-A) \cap (C-B) \\ \tilde{F}(e) \tilde{\cap} \tilde{H}(e), & if \ e\varepsilon(A-B) \cap (C-B) \\ \tilde{G}(e) \tilde{\cap} \tilde{H}(e), & if \ e\varepsilon(B-A) \cap (C-A) \\ (\tilde{F}(e) \tilde{\cap} \tilde{G}(e)) \tilde{\cap} \tilde{H}(e) ie., \ \tilde{F}(e) \tilde{\cap} (\tilde{G}(e) \tilde{\cap} \tilde{H}(e)) & if \ e\varepsilon A \cap B \cap C \end{cases}$$

[by associative property of fuzzy sets(or intuitionistic fuzzy sets).]

Now $(\tilde{L}, E)$ and $(\tilde{J}, M)$ are two fuzzy soft sets (or, intuitionistic fuzzy soft set) over the common universe, where

i) $E \subset M$ and

ii) $\forall e\varepsilon E, \tilde{L}(e)$ is a fuzzy subset (or, IF subset) of $\tilde{J}(e)$. $\Rightarrow (\tilde{L}, E)$ is a fuzzy soft subset (or, intuitionistic fuzzy soft subset) of $(\tilde{J}, M)$. (13)

Again,

i) $M \subset E$ and

ii) $\forall e\varepsilon M, \tilde{J}(e)$ is a fuzzy subset (or, IF subset) of $\tilde{L}(e)$

$\Rightarrow (\tilde{J}, M)$ is a fuzzy soft subset (or IF soft subset) of $(\tilde{L}, E)$. (14)

Combining (13) and (14),we can conclude that, $(\tilde{L}, E)$ and $(\tilde{J}, M)$ are equal FSS (or, IFSS) over the common universe. Hence proved.

5.1 Verification of this property by an example (for FSS):

Let U be the set of four ponds,say, U = $\{P_1, P_2, P_3, P_4\}$.

Let A, B, C be three sets of parameters, given by,

A = $\{ huge, average \}$

B = $\{ huge, average, poor \}$

C = $\{ average, poor \}$

Now let (F,A) denotes Area of the ponds,

(G,B) denotes Amount of water in the ponds,

(H,C) denotes Commercial benefit from the ponds

given by,

F(huge) = $\{P_1/.8, P_2/.2, P_3/.5, P_4/.4\}$

F(average) = $\{P_1/.3, P_2/.2, P_3/.9, P_4/.7\}$

G(huge) = $\{P_1/.7, P_2/.1, P_3/.4, P_4/.5\}$

G(average) = $\{P_1/.4, P_2/.3, P_3/.8, P_4/.6\}$

G(poor) = $\{P_1/.2, P_2/.6, P_3/.8, P_4/.4\}$

H(average) = $\{P_1/.2, P_2/.3, P_3/.7, P_4/.7\}$

H(poor) = $\{P_1/.3, P_2/.7, P_3/.8, P_4/.5\}$

Now consider,

$$(\tilde{F},A)\tilde{\cap}_E((\tilde{G},B)\tilde{\cap}_E(\tilde{H},C))$$
$$=(\tilde{F},A)\tilde{\cap}_E(\tilde{O},D)[\ where\ D=B\cup C=\{\ huge, average, poor\ \}$$
$$and\ \tilde{O}(huge)=\{P_1/.7,P_2/.1,P_3/.4,P_4/.5\},$$
$$\tilde{O}(\ average)=\{P_1/.2,P_2/.3,P_3/.7,P_4/.6\}$$
$$\tilde{O}(\ poor)=\{P_1/.2,P_2/.6,P_3/.8,P_4/.4\}]$$
$$=(\tilde{L},E)\ where\ E=A\cup D=\{\ huge, average, poor\ \}$$
$$and\ \tilde{L}(huge)=\{P_1/.7,P_2/.1,P_3/.4,P_4/.4\},$$
$$\tilde{L}(\ average)=\{P_1/.2,P_2/.2,P_3/.7,P_4/.6\},$$
$$\tilde{L}(\ poor)=\{P_1/.2,P_2/.6,P_3/.8,P_4/.4\}$$

Next consider,

$$((\tilde{F},A)\tilde{\cap}_E(\tilde{G},B))\tilde{\cap}_E(\tilde{H},C)$$
$$=(\tilde{I},K)\tilde{\cap}_E(\tilde{H},C)[\ where\ K=A\cup B=\{\ huge, average, poor\ \}$$
$$and\ \tilde{I}(\ huge\ )=\{P_1/.7,P_2/.1,P_3/.4,P_4/.4\},$$
$$\tilde{I}(\ average)=\{P_1/.3,P_2/.2,P_3/.8,P_4/.6\},$$
$$\tilde{I}(poor\ )=\{P_1/.2,P_2/.6,P_3/.8,P_4/.4\}]$$
$$=(\tilde{J},M)[\ where\ M=K\cup C=\{\ huge, average, poor\ \}$$
$$and\ \tilde{J}(\ huge\ )=\{P_1/.7,P_2/.1,P_3/.4,P_4/.4\}$$
$$\tilde{J}(\ average)=\{P_1/.2,P_2/.2,P_3/.7,P_4/.6\}$$
$$\tilde{J}(poor)=\{P_1/.2,P_2/.6,P_3/.8,P_4/.4\}]$$

Here $(\tilde{L},E)$ and $(\tilde{J},M)$ are two FSS which satisfy the following properties,

i) $E\subset M$ and

ii) $\forall e \varepsilon E, \tilde{L}(e)$ is a fuzzy subset of $\tilde{J}(e)$.

$\Rightarrow (\tilde{L},E)\tilde{\subseteq}(\tilde{J},M)$ (15)

Similarly, $(\tilde{J},M)\tilde{\subseteq}(\tilde{L},E)$ (16)

Combining (15) and (16) we have, $(\tilde{L},E)$ and $(\tilde{J},M)$ are fuzzy soft equal. Hence the associative property of FSS w.r.t extended intersection over the common universe is verified by this example.

5.2 Verification of this property by an example (for IFSS):

Consider example 3.2.

$\hat{F}$ (huge) = $\{C_1/(.8,.1),C_2/(.3,.5),C_3/(1,0),C_4/(.6,.2)\}$

$\hat{F}$ (average) = $\{C_1/(.6,.3),C_2/(.8,.2),C_3/(.2,.4),C_4/(.4,.2)\}$

$\hat{F}$ (low) = $\{C_1/(.2,.6),C_2/(.7,.2),C_3/(0,1),C_4/(.4,.3)\}$

$\hat{G}$ (large) = $\{C_1/(.4,.3),C_2/(.8,.1),C_3/(.9,.1),C_4/(.5,.2)\}$

$\hat{G}$ (average) = $\{C_1/(.7,.2),C_2/(.5,.3),C_3/(.1,.5),C_4/(.6,.3)\}$

$\hat{H}$ (huge) = $\{C_1/(.7,.1),C_2/(.4,.3),C_3/(.8,.1),C_4/(.5,.3)\}$

$\hat{H}$ (average) = $\{C_1/(.6,.2),C_2/(.7,.2),C_3/(.3,.6),C_4/(.5,.4)\}$

Now consider,

$$(\hat{F},A)\hat{\cap}_E((\hat{G},B)\hat{\cap}_E(\hat{H},C))$$
$$=(\hat{F},A)\hat{\cap}_E(\hat{O},D)[\ where\ D=B\cup C=\{\ large, average, huge\}$$
$$and\ \hat{O}(\ large=\{C_1/(.4,.3),C_2/(.8,.1),C_3/(.9,.1),C_4/(.5,.2)\}$$
$$\hat{O}(\ average)=\{C_1/(.6,.2),C_2/(.5,.2),C_3/(.1,.2),C_4/(.5,.3)\}$$
$$\hat{O}(\ huge)=\{C_1/(.7,.1),C_2/(.4,.3),C_3/(.8,.1),C_4/(.5,.3)\}$$
$$=(\hat{L},E)[\ where\ E=A\cup D=A\cup(B\cup C)=\{\ huge, average, low, large\}$$
$$and\ \hat{L}(\ huge)=\{C_1/(.7,.1),C_2/(.3,.5),C_3/(.8,.1),C_4/(.5,.3)\}$$
$$\hat{L}(\ average)=\{C_1/(.6,.3),C_2/(.5,.2),C_3/(.1,.4),C_4/(.4,.3)\}$$

Next consider,

$$((\hat{F},A)\hat{\cap}_E(\hat{G},B))\hat{\cap}_E(\hat{H},C)$$
$$=(\hat{I},K)\hat{\cap}_E(\hat{H},C)[\ where\ K=A\cup B=\{\ huge, average, low, large\}$$
$$and\ \hat{I}(huge)=\{C_1/(.8,.1),C_2/(.3,.5),C_3/(1,0),C_4/(.6,.2)\}$$
$$\hat{I}(\ average)=\{C_1/(.6,.3),C_2/(.5,.3),C_3/(.1,.5),C_4/(.4,.3)\}$$
$$\hat{I}(low)=\{C_1/(.2,.6),C_2/(.7,.2),C_3/(0,1),C_4/(.4,.3)\}$$
$$=(\hat{J},M)[\ where\ M=K\cup C=(A\cup B)\cup C=\{\ huge, average, low, large\}$$
$$and\ \hat{J}(\ huge)=\{C_1/(.7,.1),C_2/(.3,.5),C_3/(.8,.1),C_4/(.5,.3)\}$$
$$\hat{J}(average)=\{C_1/(.6,.3),C_2/(.5,.3),C_3/(.1,.5),C_4/(.4,.3)\}$$
$$\hat{J}(low)=\{C_1/(.2,.6),C_2/(.7,.2),C_3/(0,1),C_4/(.4,.3)\}]$$

Here $(\hat{L},E)$ and $(\hat{J},M)$ are two IFSS over the common universe where

i) $E\subset M$

ii) $\forall e\varepsilon E,\hat{L}(e)$ is a intuitionistic fuzzy subset of $\hat{J}(e)$.

$$\Rightarrow(\hat{L},E)\tilde{\subseteq}(\hat{J},M) \qquad (17)$$

[$\tilde{\subseteq}$ *denotes intuitionistic fuzzy soft subset*].

Similarly, $(\hat{J},M)\tilde{\subseteq}(\hat{L},E)$ (18)

Combining (17) and (18) we conclude that, $(\hat{L},E)$ and $(\hat{J},M)$ are IFS equal. Hence the associative property of IFSS w.r.t extended intersection over the common universe is verified by this example.

6 Associative property of FSS (or, IFSS) w.r.t restricted intersection over the common universe:

Let $(\tilde{F},A),(\tilde{G},B),(\tilde{H},C)$ are three FSS (or, IFSS). Then we have to prove,

Proof:

$$L.H.S = (\tilde{F},A)\tilde{\cap}_R((\tilde{G},B)\tilde{\cap}_R(\tilde{H},C))$$
$$= (\tilde{F},A)\tilde{\cap}_R(\tilde{O},D) \ \ where\ D = B\cap C\ and\ \forall e\varepsilon D, \tilde{O}(e) = \tilde{G}(e)\tilde{\cap}\tilde{H}(e)$$
$$= (\tilde{L},E)$$

$$where\ E = A\cap D = A\cap(B\cap C)\ and\ \forall e\varepsilon E, \tilde{L}(e) = \tilde{F}(e)\tilde{\cap}\tilde{O}(e) = \tilde{F}(e)\tilde{\cap}(\tilde{G}(e)\tilde{\cap}\tilde{H}(e))$$

$$R.H.S = ((\tilde{F},A)\tilde{\cap}_R(\tilde{G},B))\tilde{\cap}_R(\tilde{H},C)$$
$$= (\tilde{I},K)\tilde{\cap}_R(\tilde{H},C)[\ where\ K = A\cap B\ and\ \forall e\varepsilon K, \tilde{I}(e) = \tilde{F}(e)\cap\tilde{G}(e)]$$
$$= (\tilde{J},M)$$

where $M = K\cap C = (A\cap B)\cap C = A\cap(B\cap C)$ [by associative property of crisp sets] and $\forall e\varepsilon M$,

$\tilde{J}(e)$

$$= \tilde{I}(e)\tilde{\cap}\tilde{H}(e)$$
$$= (\tilde{F}(e)\tilde{\cap}\tilde{G}(e))\tilde{\cap}\tilde{H}(e)$$
$$= \tilde{F}(e)\tilde{\cap}(\tilde{G}(e)\tilde{\cap}\tilde{H}(e))$$

[*by associative property of fuzzy sets (or, IFS)*]

Now $(\tilde{L},E)$ and $(\tilde{J},M)$ are two fuzzy soft sets (or, intuitionistic fuzzy soft set) over the common universe,where

i) $E\subset M$ and

ii) $\forall e\varepsilon E, \tilde{L}(e)$ is a fuzzy subset (or, IF subset) of $\tilde{J}(e)$.

$\Rightarrow(\tilde{L},E)$ is a fuzzy soft subset (or, intuitionistic fuzzy soft subset) of $(\tilde{J},M)$

ie., $(\tilde{L},E)\tilde{\subseteq}(\tilde{J},M)$ (19)

Again, i) $M\subset E$ and

ii) $\forall e\varepsilon M, \tilde{J}(e)$ is a fuzzy subset (or, IF subset) of $\tilde{L}(e)$

$\Rightarrow(\tilde{J},M)$ is a fuzzy soft subset (or IF soft subset) of $(\tilde{L},E)$.

ie., $(\tilde{J},M)\tilde{\subseteq}(\tilde{L},E)$ (20)

Combining (19) and (20) we have, $(\tilde{L},E)$ and $(\tilde{J},M)$ are fuzzy soft equal. Hence the associative property of FSS w.r.t extended intersection over the common universe is verified by this example.

6.1 Verification of this property by an example (for FSS):

Let U be the set of four ponds,say, U = $\{P_1,P_2,P_3,P_4\}$. Let A, B, C be three sets of parameters, given by,

A = $\{ huge, average\}$

B = $\{ huge, average, poor\}$

C = $\{ average, poor \}$

Now let (F,A) denotes Area of the ponds,

(G,B) denotes Amount of water in the ponds,

(H,C) denotes Commercial benefit from the ponds

given by,

F(huge) = $\{P_1/.8,P_2/.2,P_3/.5,P_4/.4\}$

F(average) = $\{P_1/.3,P_2/.2,P_3/.9,P_4/.7\}$

G(huge) = $\{P_1/.7,P_2/.1,P_3/.4,P_4/.5\}$

G(average) = $\{P_1/.4,P_2/.3,P_3/.8,P_4/.6\}$

G(poor) = $\{P_1/.2,P_2.6,P_3/.8,P_4/.4\}$

H(average) = $\{P_1/.2,P_2/.3,P_3/.7,P_4/.7\}$

H(poor) = $\{P_1/.3,P_2.7,P_3/.8,P_4/.5\}$

Now consider,

$$(\tilde{F},A)\tilde{\cap}_R((\tilde{G},B)\tilde{\cap}_R(\tilde{H},C))$$

$$=(\tilde{F},A)\tilde{\cap}_R(\tilde{O},D)[\ where\ D=B\cap C=\{\ average, poor\ \}$$

$$and\ \tilde{O}(\ average)=\{P_1/.2,P_2/.3,P_3/.7,P_4/.6\}$$

$$\tilde{O}(\ poor)=\{P_1/.2,P_2.6,P_3/.8,P_4/.4\}]$$

$$=(\tilde{L},E)\ where\ E=A\cap D=\{\ average\}$$

$$and\ \tilde{L}(\ average)=\{P_1/.2,P_2/.2,P_3/.7,P_4/.6\}$$

Next consider,

$$((\tilde{F},A)\tilde{\cap}_R(\tilde{G},B))\tilde{\cap}_R(\tilde{H},C)$$

$$=(\tilde{I},K)\tilde{\cap}_R(\tilde{H},C)[\ where\ K=A\cap B=\{\ huge, average\}$$

$$and\ \tilde{I}(\ huge)=\{P_1/.7,P_2/.1,P_3/.4,P_4/.4\},$$

$$\tilde{I}(\ average)=\{P_1/.3,P_2/.2,P_3/.8,P_4/.6\},$$

$$=(\tilde{J},M)[\ where\ M=K\cap C=\{\ average\}$$

$and\ \tilde{J}(\ average) = \{P_1/.2, P_2/.2, P_3/.7, P_4/.6\}]$

Here $(\tilde{L}, E)$ and $(\tilde{J}, M)$ are two FSS which satisfy the following properties,
i) $E \subset M$ and
ii) $\forall e \varepsilon E, \tilde{L}(e)$ is a fuzzy subset of $\tilde{J}(e)$.
$\Rightarrow (\tilde{L}, E) \tilde{\subseteq} (\tilde{J}, M)$ (21)
Similarly, $(\tilde{J}, M) \tilde{\subseteq} (\tilde{L}, E)$ (22)
Combining (21) and (22) we have, $(\tilde{L}, E)$ and $(\tilde{J}, M)$ are fuzzy soft equal. Hence the associative property of FSS w.r.t restricted intersection over the common universe is verified by this example.

6.2 Verification of this property by an example (for IFSS):

Consider example 3.2.

$\hat{F}$ (huge) = $\{C_1/(.8,.1), C_2/(.3,.5), C_3/(1,0), C_4/(.6,.2)\}$

$\hat{F}$ (average) = $\{C_1/(.6,.3), C_2/(.8,.2), C_3/(.2,.4), C_4/(.4,.2)\}$

$\hat{F}$ (low) = $\{C_1/(.2,.6), C_2/(.7,.2), C_3/(0,1), C_4/(.4,.3)\}$

$\hat{G}$ (large) = $\{C_1/(.4,.3), C_2/(.8,.1), C_3/(.9,.1), C_4/(.5,.2)\}$

$\hat{G}$ (average) = $\{C_1/(.7,.2), C_2/(.5,.3), C_3/(.1,.5), C_4/(.6,.3)\}$

$\hat{H}$ (huge) = $\{C_1/(.7,.1), C_2/(.4,.3), C_3/(.8,.1), C_4/(.5,.3)\}$

$\hat{H}$ (average) = $\{C_1/(.6,.2), C_2/(.7,.2), C_3/(.3,.6), C_4/(.5,.4)\}$

Now consider,

$$(\hat{F}, A) \hat{\cap}_R ((\hat{G}, B) \hat{\cap}_R (\hat{H}, C))$$
$$= (\hat{F}, A) \hat{\cap}_R (\hat{O}, D)[\ where\ D = B \cap C = \{\ average\}$$
$$and\ \hat{O}(\ average) = \{C_1/(.6,.2), C_2/(.5,.3), C_3/(.1,.6), C_4/(.5,.4)\}$$
$$= (\hat{L}, E)[\ where\ E = A \cap D = A \cap (B \cap C) = \{\ average\}$$
$$and\ \hat{L}(\ average) = \{C_1/(.6,.3), C_2/(.5,.3), C_3/(.1,.6), C_4/(.4,.4)\}]$$

Next consider,

$$((\hat{F}, A) \hat{\cap}_R (\hat{G}, B)) \hat{\cap}_R (\hat{H}, C)$$
$$= (\hat{I}, K) \hat{\cap}_R (\hat{H}, C)[\ where\ K = A \cap B = \{\ average\}$$
$$and\ \hat{I}(\ average) = \{C_1/(.6,.3), C_2/(.5,.3), C_3/(.1,.5), C_4/(.4,.3)\}$$
$$= (\hat{J}, M)[\ where\ M = K \cap C = (A \cap B) \cap C = \{\ average\}$$
$$and\ \hat{J}(\ average) = \{C_1/(.6,.3), C_2/(.5,.3), C_3/(.1,.6), C_4/(.4,.4)\}]$$

Here $(\hat{L}, E)$ and $(\hat{J}, M)$ are two IFSS over the common universe where

i) $E \subset M$

ii) $\forall e \varepsilon E, \hat{L}(e)$ is a intuitionistic fuzzy subset of $\hat{J}(e)$.

$\Rightarrow (\hat{L}, E) \tilde{\subseteq} (\hat{J}, M)$ (23)

[$\tilde{\subseteq}$ denotes intuitionistic fuzzy soft subset]

Similarly, $(\hat{J}, M) \tilde{\subseteq} (\hat{L}, E)$ (24)

Combining (23) and (24) we conclude that, $(\hat{L}, E)$ and $(\hat{J}, M)$ are IFS equal.Hence the associative property of IFSS w.r.t restricted intersection over the common universe is verified by this example.

7 Distributive property of extended union of FSS (or, IFSS) w.r.t extended intersection over the common universe may not hold:

7.1 Verification of this statement by two examples :

Example 7.1:

Let U be the set of four cities, say, U = $\{C_1, C_2, C_3, C_4\}$. Let A,B,C be three sets of parameters,given by,

A = $\{ huge, average, low \}$

B = $\{ large, average \}$

C = $\{ huge, average \}$

Now let $(\tilde{F}, A)$ denotes population of the cities,

$(\tilde{G}, B)$ denotes area of the cities,

$(\tilde{H}, C)$ denotes pollution in the cities.

given by,

$\tilde{F}$ (huge) = $\{C_1/.8, C_2/.3, C_3/1, C_4/.6\}$

$\tilde{F}$ (average) = $\{C_1/.6, C_2/.8, C_3/.2, C_4/.4\}$

$\tilde{F}$ (low) = $\{C_1/.2, C_2/.7, C_3/0, C_4/.4\}$

$\tilde{G}$ (large) = $\{C_1/.4, C_2/.8, C_3/.9, C_4/.5\}$

$\tilde{G}$ (average) = $\{C_1/.7, C_2/.8, C_3/.2, C_4/.4\}$

$\tilde{H}$ (huge) = $\{C_1/.7, C_2/.2, C_3/.9, C_4/.5\}$

$\tilde{H}$ (average) = $\{C_1/.5, C_2/.7, C_3/.1, C_4/.3\}$

Now consider,

$$(\tilde{F}, A) \tilde{\cup}_E ((\tilde{G}, B) \tilde{\cap}_E (\tilde{H}, C))$$

$$= (\tilde{F}, A) \tilde{\cup}_E (\tilde{O}, D), [\ where\ D = B \cup C = \{ large, average, huge \}$$

$$\text{and } \tilde{O}(large) = \{C_1/.4, C_2/.8, C_3/.9, C_4/.5\}$$
$$\tilde{O}(average) = \{C_1/.5, C_2/.7, C_3/.1, C_4/.3\}$$
$$\tilde{O}(huge) = \{C_1/.7, C_2/.2, C_3/.9, C_4/.5\}$$
$$= (\tilde{L}, E)[\text{ where } E = A \cup D = A \cup (B \cup C) = \{ large, average, huge, low \}$$
$$\text{and } \tilde{L}(large) = \{C_1/.4, C_2/.8, C_3/.9, C_4/.5\}$$
$$\tilde{L}(average) = \{C_1/.6, C_2/.8, C_3/.2, C_4/.4\}$$
$$\tilde{L}(huge) = \{C_1/.8, C_2/.3, C_3/1, C_4/.6\}$$
$$\tilde{L}(low) = \{C_1/.2, C_2/.7, C_3/0, C_4/.4\}]$$

Next consider,

$$((\tilde{F}, A) \tilde{\cup}_E (\tilde{G}, B)) \tilde{\cap}_E ((\tilde{F}, A) \tilde{\cup}_E (\tilde{H}, C))$$
$$= (\tilde{I}, K) \tilde{\cap}_E (\tilde{J}, M)[\text{ where } K = A \cup B = \{ huge, average, low, large \},$$
$$M = A \cup C = \{ huge, average, low \}$$
$$\tilde{I}(huge) = \{C_1/.8, C_2/.3, C_3/1, C_4/.6\}$$
$$\tilde{I}(average) = \{C_1/.7, C_2/.8, C_3/.2, C_4/.4\}$$
$$\tilde{I}(low) = \{C_1/.2, C_2/.7, C_3/0, C_4/.4\}$$
$$\tilde{I}(large) = \{C_1/.4, C_2/.8, C_3/.9, C_4/.5\},$$
$$\text{and } \tilde{J}(huge) = \{C_1/.8, C_2/.3, C_3/1, C_4/.6\}$$
$$\tilde{J}(average) = \{C_1/.6, C_2/.8, C_3/.2, C_4/.4\}$$
$$\tilde{J}(low) = \{C_1/.2, C_2/.7, C_3/0, C_4/.4\}]$$
$$= (\tilde{N}, P)[\text{ where} P = K \cup M = \{ large, average, huge, low \} \text{and}$$
$$\tilde{N}(large) = \{C_1/.4, C_2/.8, C_3/.9, C_4/.5\},$$
$$\tilde{N}(average) = \{C_1/.6, C_2/.8, C_3/.2, C_4/.4\}$$
$$\tilde{N}(huge) = \{C_1/.8, C_2/.3, C_3/1, C_4/.6\}$$
$$\tilde{N}(low) = \{C_1/.2, C_2/.7, C_4/.4\}$$

Here it is clear that $(\tilde{L}, E)$ is a FS subset of $(\tilde{N}, P)$ and viceversa.Therefore $(\tilde{L}, E)$ and $(\tilde{N}, P)$ are two equal FSS.

Therefore here $(\tilde{F}, A) \tilde{\cup}_E ((\tilde{G}, B) \tilde{\cap}_E (\tilde{H}, C)) = ((\tilde{F}, A) \tilde{\cup}_E (\tilde{G}, B)) \tilde{\cap}_E ((\tilde{F}, A) \tilde{\cup}_E (\tilde{H}, C))$. Hence the distributive property of FSS w.r.t union over the common universe holds in this example.

Example 7.2:

Let U be the set of four dresses, say, U = $\{d_1, d_2, d_3, d_4\}$. Let A,B,C be three sets of parameters,given by,

A = $\{ high, cheap \}$

B = $\{ high, cheap, assured \}$
C = $\{ high, assured \}$
Now let $(\tilde{F}, A)$ denotes cost of the dresses,
$(\tilde{G}, B)$ denotes quality of the dresses,
$(\tilde{H}, C)$ denotes attractiveness of the dresses,
given by,
$\tilde{F}$ (high) = $\{d_1/.8, d_2/.5, d_3/.2, d_4/.6\}$
$\tilde{F}$ (cheap) = $\{d_1/.2, d_2/.3, d_3/.4, d_4/.5\}$
$\tilde{G}$ (high) = $\{d_1/.8, d_2/.3, d_3/.4, d_4/.5\}$
$\tilde{G}$ (cheap) = $\{d_1/.1, d_2/.4, d_3/.3, d_4/.6\}$
$\tilde{G}$ (assured) = $\{d_1/.9, d_2/.5, d_3/.3, d_4/.7\}$
$\tilde{H}$ (high) = $\{d_1/.3, d_2/.5, d_3/.7, d_4/.8\}$
$\tilde{H}$ (assured) = $\{d_1/.2, d_2/.5, d_3/.8, d_4/.8\}$
Now consider,

$$(\tilde{F}, A) \tilde{\cup}_E ((\tilde{G}, B) \tilde{\cap}_E (\tilde{H}, C))$$
$$= (\tilde{F}, A) \tilde{\cup}_E (\tilde{O}, D)[where\ D = B \cup C = \{ high, cheap, assured \}$$
$$and \tilde{O}(high) = \{d_1/.3, d_2/.5, d_3/.7, d_4/.8\}$$
$$\tilde{O}(cheap) = \{d_1/.1, d_2/.4, d_3/.3, d_4/.6\}$$
$$\tilde{O}(assured) = \{d_1/.2, d_2/.5, d_3/.3, d_4/.7\}]$$
$$= (\tilde{L}, E)\ where E = A \cup D = A \cup (B \cap C) = \{ high, cheap, assured \}$$
$$and\ \tilde{L}(high) = \{d_1/.8, d_2/.5, d_3/.7, d_4/.8\}$$
$$\tilde{L}(cheap) = \{d_1/.2, d_2/.4, d_3/.4, d_4/.6\}$$
$$\tilde{L}(assured) = \{d_1/.2, d_2/.5, d_3/.3, d_4/.7\}$$

Next consider,

$$((\tilde{F}, A) \tilde{\cup}_E (\tilde{G}, B)) \tilde{\cap}_E ((\tilde{F}, A) \tilde{\cup}_E (\tilde{H}, C))$$
$$= (\tilde{I}, K) \tilde{\cap}_E (\tilde{J}, M)[\ where\ K = A \cup B = \{ high, cheap, assured \},$$
$$M = A \cup C = \{ high, cheap, assured \}$$
$$\tilde{I}(high) = \{d_1/.8, d_2/.5, d_3/.4, d_4/.6\}$$
$$\tilde{I}(cheap) = \{d_1/.2, d_2/.4, d_3/.4, d_4/.6\}$$
$$\tilde{I}(assured) = \{d_1/.9, d_2/.5, d_3/.3, d_4/.7\}$$
$$and \tilde{J}(high) = \{d_1/.8, d_2/.5, d_3/.7, d_4/.8\}$$
$$\tilde{J}(cheap) = \{d_1/.2, d_2/.3, d_3/.4, d_4/.5\}$$
$$\tilde{J}(assured) = \{d_1/.2, d_2/.5, d_3/.8, d_4/.8\}]$$

$= (\tilde{N}, P)[\ where P = K \cup M = \{\ high, cheap, assured\ \}$
$and \tilde{N}(high) = \{d_1/.8, d_2/.5, d_3/.4, d_4/.6\}$
$\tilde{N}(cheap) = \{d_1/.2, d_2/.3, d_3/.4, d_4/.5\}$
$\tilde{N}(assured) = \{d_1/.2, d_2/.5, d_3/.3, d_4/.7\}]$

Now since $(\tilde{L}, E)$ and $(\tilde{N}, P)$ are two different fuzzy soft sets therefore here
$(\tilde{F}, A)\tilde{\cup}_E ((\tilde{G}, B)\tilde{\cap}_E (\tilde{H}, C)) \neq ((\tilde{F}, A)\tilde{\cup}_E (\tilde{G}, B))\tilde{\cap}_E ((\tilde{F}, A)\tilde{\cup}_E (\tilde{H}, C))$
Hence the distributive property of extended union of FSS w.r.t extended intersection over the common universe does not hold in this example.

8 Distributive property of restricted union of FSS (or, IFSS) w.r.t restricted intersection over the common universe:

Let $(\tilde{F}, A), (\tilde{G}, B), (\tilde{H}, C)$ are three FSS(or, IFSS). Then we have to prove,
$(\tilde{F}, A)\tilde{\cup}_R ((\tilde{G}, B)\tilde{\cap}_R (\tilde{H}, C)) = ((\tilde{F}, A)\tilde{\cup}_R (\tilde{G}, B))\tilde{\cap}_R ((\tilde{F}, A)\tilde{\cup}_R (\tilde{H}, C))$.
Proof:

$L.H.S. = (\tilde{F}, A)\tilde{\cup}_R ((\tilde{G}, B)\tilde{\cap}_R (\tilde{H}, C))$
$= (\tilde{F}, A)\tilde{\cup}_R (\tilde{O}, D)[\ where\ D = B \cap C\ and\ \forall e \varepsilon D, \tilde{O}(e) = \tilde{G}(e)\tilde{\cap}\tilde{H}(e)],$
$= (\tilde{L}, E)\ where\ E = A \cap D = A \cap B \cap C\ and$

$\tilde{L}(e) = \tilde{F}(e)\tilde{\cup}(\tilde{G}(e)\tilde{\cap}\tilde{H}(e)), \forall e \varepsilon A \cap B \cap C.$

$R.H.S. = ((\tilde{F}, A)\tilde{\cup}_R (\tilde{G}, B))\tilde{\cap}_R ((\tilde{F}, A)\tilde{\cup}_R (\tilde{H}, C))$
$= (\tilde{I}, K)\cap_R (\tilde{J}, M)[\ where\ K$

$= A \cap B, M = A \cap C, \tilde{I}(e) = \tilde{F}(e)\tilde{\cup}\tilde{G}(e), \forall e \varepsilon A \cap B\ and\ \tilde{J}(e) = \tilde{F}(e)\tilde{\cup}H(e), \forall e \varepsilon A \cap C]$
$= (\tilde{N}, P)$
where P $= K \cap M = (A \cap B) \cap (A \cap C) = A \cap B \cap C$
and $\tilde{N}(e) = \tilde{I}(e)\tilde{\cap}\tilde{J}(e), \forall e \varepsilon A \cap B \cap C$
ie., N(e)
$= (\tilde{F}(e)\tilde{\cup}\tilde{G}(e))\tilde{\cap}(\tilde{F}(e)\tilde{\cup}\tilde{H}(e))$

$= \tilde{F}(e)\tilde{\cup}(\tilde{G}(e)\tilde{\cap}\tilde{H}(e)), \forall e \varepsilon A \cap B \cap C[by\ the\ distributive\ property of\ FS(\ or\ IFS\ )]$

Here$(\tilde{L}, E)$ and $(\tilde{N}, P)$ are two FSS (or IFSS) which satisfy,

i) $E \subset P$ and

ii) $\forall e \varepsilon E, \tilde{L}(e)$ is a fuzzy subset (or intuitiontionistic fuzzy subset) of $\tilde{J}(e)$.

$\Rightarrow (\tilde{L}, E) \tilde{\subset} (\tilde{N}, P)$ (25)

[$\tilde{\subset}$ denotes FS(or IFS) subset]

Similarly, we can prove, $(\tilde{N}, P) \tilde{\subset} (\tilde{L}, E)$ (26)

Combining (25) and (26) we get, $(\tilde{L}, E)$ and $(\tilde{N}, P)$ are two equal FSS (or IFSS) over the common universe. Hence proved.

8.1 Verification of this property by an example (for FSS):

Consider example 7.1. Now consider,

$$(\tilde{F}, A) \tilde{\cup}_R ((\tilde{G}, B) \tilde{\cap}_R (\tilde{H}, C))$$

$$= (\tilde{F}, A) \tilde{\cup}_R (\tilde{O}, D), [\ where\ D = B \cap C = \{\ average\}$$

$$and\ \tilde{O}(average) = \{C_1/.5, C_2/.7, C_3/.1, C_4/.3\}$$

$$= (\tilde{L}, E) [\ where\ E = A \cap D = A \cap (B \cap C) = \{\ average\}$$

$$and\ \tilde{L}(average) = \{C_1/.6, C_2/.8, C_3/.2, C_4/.4\}]$$

Next consider,

$$((\tilde{F}, A) \tilde{\cup}_R (\tilde{G}, B)) \tilde{\cap}_R ((\tilde{F}, A) \tilde{\cup}_R (\tilde{H}, C))$$

$$= (\tilde{I}, K) \tilde{\cap}_R (\tilde{J}, M) [\ where\ K = A \cap B = \{\ average\},$$

$$M = A \cap C = \{\ average\}$$

$$\tilde{I}(average) = \{C_1/.7, C_2/.8, C_3/.2, C_4/.4\}$$

$$and\ \tilde{J}(average) = \{C_1/.6, C_2/.8, C_3/.2, C_4/.4\}$$

$$= (\tilde{N}, P) [\ where P = K \cap M = \{\ average\}\ and$$

$$\tilde{N}(average) = \{C_1/.6, C_2/.8, C_3/.2, C_4/.4\}]$$

Clearly, $(\tilde{L}, E)$ and $(\tilde{J}, M)$ are fuzzy soft equal. Hence the distributive property of restricted union of FSS (or, IFSS) w.r.t restricted intersection over the common universe is verified by this example.

8.2 Verification of this property by an example (for IFSS):

Let U be the set of four dresses, say, U = $\{d_1, d_2, d_3, d_4\}$. Let A,B,C be three sets of parameters, given by,

A = $\{\ high, cheap\ \}$

B = $\{\ high, cheap, assured\ \}$

C = $\{\ high, assured\ \}$

Now let $(\tilde{F}, A)$ denotes cost of the dresses,

$(\tilde{G}, B)$ denotes quality of the dresses,

$(\tilde{H}, C)$ denotes attractiveness of the dresses,

given by,

$\tilde{F}$ (high) = $\{d_1/(.8,,1), d_2/(.5,.2), d_3/(.2,.4), d_4/(.6,.3)\}$

$\tilde{F}$ (cheap) = $\{d_1/(.2,.7), d_2/(.3,.3), d_3/(.4,.3), d_4/(.5,.3)\}$

$\tilde{G}$ (high) = $\{d_1/(.8,.1), d_2/(.3,.6), d_3/(.4,.3), d_4/(.5,.3)\}$

$\tilde{G}$ (cheap) = $\{d_1/(.1,.7), d_2/(.4,.3), d_3/(.3,.7), d_4/(.6,.2)\}$

$\tilde{G}$ (assured) = $\{d_1/(.9,.1), d_2/(.5,.3), d_3/(.3,.5), d_4/(.7,.2)\}$

$\tilde{H}$ (high) = $\{d_1/(.3,.5), d_2/(.5,.2), d_3/(.7,.2), d_4/(.8,.1)\}$

$\tilde{H}$ (assured) = $\{d_1/(.2,.6), d_2/(.5,.3), d_3/(.8,.1), d_4/(.8,0)\}$

Now consider,

$$(\hat{F}, A)\hat{\cup}_R ((\hat{G}, B)\hat{\cap}_R (\hat{H}, C))$$

$$= (\hat{F}, A)\hat{\cup}_R (\hat{O}, D),[\ where\ D = B \cap C = \{\ high, assured\ \}$$

$$and \hat{O}(high\ ) = \{d_1/(.3,.5), d_2/(.3,.6), S_3/(.4,.3), S_4/(.5,.3)\}$$

$$\hat{O}(\ assured\ ) = \{d_1/(.2,.6), d_2/(.5,.3), d_3/(.3,.5), d_4/(.7,.2)\}]$$

$$= (\hat{L}, E)\ where E = A \cap D = A \cap (B \cap C) = \{\ high\ \}$$

$$and\ \hat{L}(high) = \{d_1/(.8,.1), d_2/(.5,.2), d_3/(.4,.3), d_4/(.6,.3)\}$$

Next consider,

$$((\hat{F}, A)\tilde{\cup}_R (\hat{G}, B))\tilde{\cap}_R ((\hat{F}, A)\tilde{\cup}_R (\hat{H}, C))$$

$$= (\hat{I}, K)\tilde{\cap}_R (\hat{J}, M)[\ where\ K = A \cap B = \{\ high, cheap\ \},$$

$$M = A \cap C = \{\ high, assured\ \}$$

$$\hat{I}(\ high) = \{d_1/(.8,.1), d_2/(.5,.2), d_3/(.4,.3), d_4/(.6,.3)\}$$

$$\hat{I}(cheap) = \{d_1/(.2,.7), d_2/(.4,.3), d_3/(.4,.3), d_4/(.6,.2)\}$$

$$and \hat{J}(high) = \{d_1/(.8,.1), d_2/(.5,.2), d_3/(.7,.2), d_4/(.8,.1)\}$$

$$\hat{J}(\ assured\ ) = \{d_1/(.2,.6), d_2/(.5,.3), d_3/(.8,.1), d_4/(.8,0)\}]$$

$$= (\hat{N}, P)[\ where P = K \cap M = \{\ high\ \}$$

$$and \hat{N}(high) = \{d_1/(.8,.1), d_2/(.5,.2), d_3/(.4,.3), d_4/(.6,.3)\}]$$

Now since $(\hat{L}, E)$ is an intuitionistic fuzzy soft subset of $(\hat{N}, P)$ and vice-versa, therefore $(\hat{L}, E)$ and $(\hat{N}, P)$ are two equal IFSS.Hence the distributive property of restricted union of IFSS w.r.t restricted intersection over the common universe is verified by this example.

9 Distributive property of extended intersection w.r.t extended union over the common universe may not hold for all FSS (or IFSS):

Verification of this statement by two examples :

Example 9.1:

Consider the example 7.1. Now consider,

$$(\tilde{F},A)\tilde{\cap}_E((\tilde{G},B)\tilde{\cup}_E(\tilde{H},C))$$
$$=(\tilde{F},A)\tilde{\cap}_E(\tilde{O},D),[\ where\ D=B\cup C=\{\ large, average, huge\ \}$$
$$and\ \tilde{O}(\ large)=\{C_1/.4,C_2/.8,C_3/.9,C_4/.5\}$$
$$\tilde{O}(average)=\{C_1/.7,C_2/.8,C_3/.2,C_4/.4\}$$
$$\tilde{O}(\ huge)=\{C_1/.8,C_2/.3,C_3/1,C_4/.6\}$$
$$=(\tilde{L},E)[\ where\ E=A\cup D=A\cup(B\cup C)=\{\ large, average, huge, low\ \}$$
$$and\ \tilde{L}(large)=\{C_1/.4,C_2/.8,C_3/.9,C_4/.5\}$$
$$L(average)=\{C_1/.6,C_2/.8,C_3/.2,C_4/.4\}$$
$$\tilde{L}(huge)=\{C_1/.8,C_2/.3,C_3/1,C_4/.6\}$$
$$\tilde{L}(\ low)=\{C_1/.2,C_2/.7,C_3/0,C_4/.4\}]$$

Next consider,

$$((\tilde{F},A)\tilde{\cap}_E(\tilde{G},B))\tilde{\cup}_E((\tilde{F},A)\tilde{\cap}_E(\tilde{H},C))$$
$$=(\tilde{I},K)\tilde{\cup}_E(\tilde{J},M)[\ where\ K=A\cup B=\{\ huge, average, low, large\ \},$$
$$M=A\cup C=\{\ huge, average, low\ \}$$
$$\tilde{I}(huge)=\{C_1/.8,C_2/.3,C_3/1,C_4/.6\}$$
$$\tilde{I}(average)=\{C_1/.6,C_2/.8,C_3/.2,C_4/.4\}$$
$$\tilde{I}(low)=\{C_1/.2,C_2/.7,C_3/0,C_4/.4\}$$
$$\tilde{I}(large)=\{C_1/.4,C_2/.8,C_3/.9,C_4/.5\},$$
$$and\ \tilde{J}(\ huge\ )=\{C_1/.8,C_2/.3,C_3/1,C_4/.6\}$$
$$\tilde{J}(average)=\{C_1/.6,C_2/.8,C_3/.2,C_4/.4\}$$
$$\tilde{J}(low)=\{C_1/.2,C_2/.7,C_3/0,C_4/.4\}]$$
$$=(\tilde{N},P)[\ whereP=K\cup M=\{\ huge, average, low, large\ \}and$$
$$\tilde{N}(huge\ )=\{C_1/.8,C_2/.3,C_3/1,C_4/.6\}$$
$$\tilde{N}(average)=\{C_1/.6,C_2/.8,C_3/.2,C_4/.4\}$$
$$\tilde{N}(\ low\ )=\{C_1/.2,C_2/.7,C_4/.4\}$$
$$\tilde{N}(large)=\{C_1/.4,C_2/.8,C_3/.9,C_4/.5\},$$

Here it is clear that $(\tilde{L},E)$ is a FS subset of $(\tilde{N},P)$ and viceversa.Therefore $(\tilde{L},E)$ and $(\tilde{N},P)$ are two equal FSS. Hence the distributive property of extended intersection w.r.t extended union over the the common universe holds for these fuzzy soft sets.

Example 9.2:

Consider the example 8.2. Now consider,

$$(\hat{F},A)\hat{\cap}_E((\hat{G},B)\hat{\cup}_E(\hat{H},C))$$
$$=(\hat{F},A)\hat{\cap}_E(\hat{O},D),[\ where\ D=B\cup C=\{\ high, cheap, assured\}$$
$$and\ \hat{O}(high)=\{d_1/(.8,.1),d_2/(.5,.2),d_3/(.7,.2),d_4/(.8,.1)\}$$
$$\hat{O}(cheap)=\{d_1/(.1,.7),d_2/(.4,.3),d_3/(.3,.7),d_4/(.6,.2)\}$$
$$\hat{O}(\ assured\ )=\{d_1/(.9,.1),d_2/(.5,.3),d_3/(.8,.1),d_4/(.8,0)\}]$$
$$=(\hat{L},E)[\ whereE=A\cup D=A\cup(B\cap C)=\{\ high, cheap, assured\}$$
$$and\ \hat{L}(high)=\{d_1/(.8,.1),d_2/(.5,.2),d_3/(.2,.4),d_4/(.6,.3)\}$$
$$\hat{L}(\ cheap\ )=\{d_1/(.1,.6),d_2/(.4,.3),d_3/(.3,.2),d_4/(.5,.3)\}$$
$$\hat{L}(\ assured\ )=\{d_1/(.9,.1),d_2/(.5,.3),d_3/(.8,.1),d_4/(.8,0)\}]$$

Next consider,

$$((\hat{F},A)\hat{\cap}_E(\hat{G},B))\hat{\cup}_E((\hat{F},A)\hat{\cap}_E(\hat{H},C))$$
$$=(\hat{I},K)\hat{\cup}_E(\hat{J},M)[\ where\ K=A\cup B=\{\ high, cheap, assured\},$$
$$M=A\cup C=\{\ high, cheap, assured\}$$
$$\hat{I}(\ high)=\{d_1/(.8,.1),d_2/(.3,.6),d_3/(.2,.4),d_4/(.5,.3)\}$$
$$\hat{I}(cheap)=\{d_1/(.1,.7),d_2/(.3,.3),d_3/(.3,.7),d_4/(.5,.3)\}$$
$$\hat{I}(assured)=\{d_1/(.9,.1),d_2/(.5,.3),d_3/(.3,.5),d_4/(.7,.2)\}$$
$$and\ \hat{J}(high)=\{d_1/(.3,.5),d_2/(.5,.2),d_3/(.2,.4),d_4/(.6,.3)\}$$
$$\hat{J}(cheap)=\{d_1/(.2,.7),d_2/(.3,.3),d_3/(.4,.3),d_4/(.5,.3)\}$$
$$\hat{J}(\ assured\ )=\{d_1/(.2,.6),d_2/(.5,.3),d_3/(.8,.1),d_4/(.8,0)\}]$$
$$=(\hat{N},P)[\ whereP=K\cup M=\{\ high, cheap, assured\}$$
$$and\ \hat{N}(high)=\{d_1/(.8,.1),d_2/(.5,.2),d_3/(.2,.4),d_4/(.6,.3)\}$$
$$\hat{N}(cheap)=\{d_1/(.2,.7),d_2/(.3,.3),d_3/(.4,.3),d_4/(.5,.3)\}$$
$$\hat{N}(assured)=\{d_1/(.9,.1),d_2/(.5,.3),d_3/(.8,.1),d_4/(.8,0)\}]$$

Now since $(\hat{L},E)$ and $(\hat{N},P)$ are two different intuitionistic fuzzy soft sets therefore here $(\hat{F},A)\hat{\cap}_E((\hat{G},B)\hat{\cup}_E(\hat{H},C))\neq((\hat{F},A)\hat{\cap}_E(\hat{G},B))\hat{\cup}_E((\hat{F},A)\hat{\cap}_E(\hat{H},C))$.

Hence the distributive property of extended intersection w.r.t extended union over the common universe does not hold for these intuitionistic fuzzy soft sets.

10 Distributive property of restricted intersection of FSS (or, IFSS) w.r.t restricted union over the common universe:

Let $(\tilde{F},A),(\tilde{G},B),(\tilde{H},C)$ are three FSS(or, IFSS). Then we have to prove,
$(\tilde{F},A)\tilde{\cap}_R((\tilde{G},B)\tilde{\cup}_R(\tilde{H},C)) = ((\tilde{F},A)\tilde{\cap}_R(\tilde{G},B))\tilde{\cup}_R((\tilde{F},A)\tilde{\cap}_R(\tilde{H},C))$.
Proof:

$$L.H.S. = (\tilde{F},A)\tilde{\cap}_R((\tilde{G},B)\tilde{\cup}_R(\tilde{H},C))$$
$$= (\tilde{F},A)\tilde{\cap}_R(\tilde{O},D)[\ where\ D = B\cap C\ and\ \forall e\varepsilon D, \tilde{O}(e) = \tilde{G}(e)\tilde{\cup}\tilde{H}(e)]$$

$$= (\tilde{L},E)\ where\ E = A\cap D = A\cap(B\cap C)\ and$$

$$\tilde{L}(e) = \tilde{F}(e)\tilde{\cap}(\tilde{G}(e)\tilde{\cup}\tilde{H}(e)), \forall e\varepsilon A\cap B\cap C.$$

$$R.H.S. = ((\tilde{F},A)\tilde{\cap}_R(\tilde{G},B))\tilde{\cup}_R((\tilde{F},A)\tilde{\cap}_R(\tilde{H},C))$$
$$= (\tilde{I},K)\cup_R(\tilde{J},M)[\ where\ K$$

$$= A\cap B, M = A\cap C, \tilde{I}(e) = \tilde{F}(e)\tilde{\cap}\tilde{G}(e), \forall e\varepsilon A\cap B\ and\ \tilde{J}(e) = \tilde{F}(e)\tilde{\cap}\tilde{H}(e), \forall e\varepsilon A\cap C]$$
$$= (\tilde{N},P)$$
where P $= K\cap M = (A\cap B)\cap(A\cap C) = A\cap(B\cap C)$ [by the distributive property of crisp sets].
and $\tilde{N}(e) = \tilde{I}(e)\tilde{\cup}\tilde{J}(e), \forall e\varepsilon A\cap(B\cap C)$

$$ie., \tilde{N}(e) = (\tilde{F}(e)\tilde{\cap}\tilde{G}(e))\tilde{\cup}(\tilde{F}(e)\tilde{\cap}\tilde{H}(e))$$
$$= \tilde{F}(e)\tilde{\cap}(\tilde{G}(e)\tilde{\cup}\tilde{H}(e)), \forall e\varepsilon A\cap(B\cap C)$$

[by the distributive property of FS(or, IFS)].
Here$(\tilde{L},E)$ and $(\tilde{N},P)$ are two FSS (or IFSS) which satisfy,
i) $E\subset P$ and
ii) $\forall e\varepsilon E, \tilde{L}(e)$ is a fuzzy subset (or intuitiontionistic fuzzy subset) of $\tilde{J}(e)$.
$\Rightarrow (\tilde{L},E)\tilde{\subseteq}(\tilde{N},P)$ (27)
[$\tilde{\subseteq}$ denotes FS(or IFS) subset].
Similarly, we can prove, $(\tilde{N},P)\tilde{\subseteq}(\tilde{L},E)$ (28)
Combining (27) and (28) we get, $(\tilde{L},E)$ and $(\tilde{N},P)$ are two equal FSS (or IFSS) over the common universe. Hence proved.

10.1 Verification of this property by an example(for FSS):

Consider the example 7.1.
Now consider,

$$(\tilde{F},A)\tilde{\cap}_R((\tilde{G},B)\tilde{\cup}_R(\tilde{H},C))$$
$$= (\tilde{F},A)\tilde{\cap}_R(\tilde{O},D),[\ where\ D = B\cap C = \{\ average\}$$

$$and\ \tilde{O}(average) = \{C_1/.7, C_2/.8, C_3/.2, C_4/.4\}]$$
$$= (\tilde{L}, E)[\ where\ E = A \cap D = A \cap (B \cap C) = \{\ average\}$$
$$and\ \tilde{L}(average) = \{C_1/.6, C_2/.8, C_3/.2, C_4/.4\}]$$

Next consider,

$$((\tilde{F}, A)\tilde{\cap}_R (\tilde{G}, B))\tilde{\cup}_R ((\tilde{F}, A)\tilde{\cap}_R (\tilde{H}, C))$$
$$= (\tilde{I}, K)\tilde{\cup}_R (\tilde{J}, M)[\ where\ K = A \cap B = \{\ average\},$$
$$M = A \cap C = \{\ huge, average\}$$
$$\tilde{I}(average) = \{C_1/.6, C_2/.8, C_3/.2, C_4/.4\}$$
$$and\ \tilde{J}(\ huge\ ) = \{C_1/.8, C_2/.3, C_3/1, C_4/.6\}$$
$$\tilde{J}(average) = \{C_1/.6, C_2/.8, C_3/.2, C_4/.4\}]$$
$$= (\tilde{N}, P)[\ whereP = K \cap M = \{\ average\}and$$
$$\tilde{N}(average) = \{C_1/.6, C_2/.8, C_3/.2, C_4/.4\}]$$

Now since $(\tilde{L}, E)$ is an fuzzy soft subset of $(\tilde{N}, P)$ and vice-versa,therefore $(\tilde{L}, E)$ and $(\tilde{N}, P)$ are two equal FSS. Hence the distributive property of restricted intersection of FSS w.r.t restricted union over the common universe is verified by this example.

10.2 Verification of this property by an example (for IFSS):

Consider the example 8.2. Now consider,

$$(\hat{F}, A)\hat{\cap}_R ((\hat{G}, B)\hat{\cup}_R (\hat{H}, C))$$
$$= (\hat{F}, A)\hat{\cap}_R (\hat{O}, D)[\ where\ D = B \cap C = \{\ high, assured\ \}$$
$$and\hat{O}(high\ ) = \{d_1/(.8,.1), d_2/(.5,.2), d_3/(.7,.2), d_4/(.8,.1)\}$$
$$\hat{O}(\ assured\ ) = \{d_1/(.9,.1), d_2/(.5,.3), d_3/(.8,.1), d_4/(.8,0)\}]$$
$$= (\hat{L}, E)[\ whereE = A \cap D = A \cap (B \cap C) = \{\ high\ \}$$
$$and\ \hat{L}(high) = \{d_1/(.8,.1), d_2/(.5,.2), d_3/(.2,.4), d_4/(.6,.3)\}]$$

Next consider,

$$((\hat{F}, A)\hat{\cap}_R (\hat{G}, B))\hat{\cup}_R ((\hat{F}, A)\hat{\cap}_R (\hat{H}, C))$$
$$= (\hat{I}, K)\hat{\cup}_R (\hat{J}, M)[\ where\ K = A \cap B = \{\ high, cheap\ \},$$
$$M = A \cap C = \{\ high\ \}$$
$$\hat{I}(\ high) = \{d_1/(.8,.1), d_2/(.3,.6), d_3/(.2,.4), d_4/(.5,.3)\}$$
$$\hat{I}(cheap) = \{d_1/(.1,.7), d_2/(.3,.3), d_3/(.3,.7), d_4/(.5,.3)\}$$
$$and\hat{J}(high) = \{d_1/(.3,.5), d_2/(.5,.2), d_3/(.2,.4), d_4/(.6,.3)\}]$$
$$= (\hat{N}, P)[\ whereP = K \cap M = \{\ high\ \}$$

$$and\,\hat{N}(high) = \{d_1/(.8,.1), d_2/(.5,.2), d_3/(.2,.4), d_4/(.6,.3)\}]$$

Now since $(\hat{L}, E)$ is an intuitionistic fuzzy soft subset of $(\hat{N}, P)$ and vice-versa,therefore $(\hat{L}, E)$ and $(\hat{N}, P)$ are two equal IFSS. Hence the distributive property of restricted intersection of IFSS w.r.t restricted union over the common universe is verified by this example.

11 Conclusion:

For handling real life ambiguous situations we need to have methodologies which provide some form or other flexible information processing capacity. The soft set theory of Maji [18] offers the associative and distributive properties (w.r.t. union and intersection) of soft sets over the common universe. As a tentative and partial attempt towards a further development in this paper we proved the same properties for fuzzy soft sets and intuitionistic fuzzy soft sets with respect to some new operations and verified these properties by appropriate numerical examples. These properties may be used in real life problems, like decision making problem, inventory Control problem, etc.

References:

- P. K. Maji, R. Biswas and A. R. Roy, Fuzzy Soft Sets, The Journal of Fuzzy Mathematics Vol. 9, No. 3, (2001), 589-602.
- C. Nahak, Fuzzy Sets and Fuzzy Optimizations, Fuzzy sets and its applications ORSI Pre-conference workshop on OR as competitive edge, IIT, Kharagpur, January 3-4 (2007), 1-13.
- P. K. Maji, Some Aspects of Soft Sets, Fuzzy Soft Sets and Intuitionistic Fuzzy Soft Sets, Ph. D. Thesis, IIT, Kharagpur.
- M. I. Ali, F. Feng, X. Liu, W. K. Min, M. Shabir, On Some New Operations in Soft Set Theory, Computers and Mathematics with Applications, (2008), 1-7.
- D.Molodtsov,Soft Set Theory-First Results,Computers and Mathematics with Applications, 37, (1999), 19-31.
- Z.Pawlak,Rough Sets,International Journal of Information and Computer Sciences, 11,(1982), 341-356.

- H.J.Zimmerman, Fuzzy Set Theory And Its Applications, Kluwer Academic Publishers(Boston) 1996.
- L.A.Zadeh,Fuzzy Sets,Infor.And Control, 8 (1965), 338-353.
- H. Prade and D. Dubois, Fuzzy Sets and Systems Theory and Applications, Academic Press London, 1980.
- P. K. Maji, R. Biswas and A. R. Roy, An application of soft sets in a decision making problem, Computers And Mathematics with Applications 44 (2002), 1077-1083.
- K. Atanassov, Intuitionistic fuzzy sets, Fuzzy Sets and Systems, 20 (1986), 87-96.
- K. Atanassov, Operators over interval valued Intuitionistic fuzzy sets, Fuzzy Sets and Systems, 64 (1994), 159-174.
- M. B. Gorzalzany, A method of inference in approximate reasoning based on interval-valued fuzzy sets and systems, 21 (1987), 1-17.
- W. L. Gau and D. J. Buehrer, Vague Sets, IEEE Trans. System Man Cybernet, 23(2), (1993), 610-614.
- P. K. Maji, A. R. Roy, A Fuzzy Soft Set Theoretic Approach to Decision Making Problems, Journal of Computational and Applied Mathematics, 203 (2007), 412, - 418.
- P. K. Maji, R. Biswas and A. R. Roy, On Intuitionistic Fuzzy Soft Sets, The Journal of Fuzzy Mathematics, Vol. 12, No. 3, (2004), 669-683.
- P. K. Maji, R. Biswas and A. R. Roy, Intuitionistic Fuzzy Soft Sets, The Journal of Fuzzy Mathematics Vol. 9, No. 3, (2001), 677-692.
- P. K. Maji, R. Biswas and A. R. Roy, Soft Set Theory, An application of soft sets in a decision making problem, Computers And Mathematics with Applications, 45 (2003), 555-562.
- T. Mitra Basu, N. K. Mahapatra and S. K. Mondal, On Some New Operations in Fuzzy Soft Set And Intuitionistic Fuzzy Soft Set Theory, Communicated paper.